

Mathematical Induction

Part Two

Outline for Today

- ***Variations on Induction***
 - Starting later, taking different step sizes, and more!
- ***Complete Induction***
 - When one assumption isn't enough!

Recap from Last Time

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true...

$P(0)$ is true

...and it stays true...

and

$\forall k \in \mathbb{N}. (P(k) \rightarrow P(k+1))$

then

$\forall n \in \mathbb{N}. P(n)$

...then it's always true.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: Let $P(n)$ be the statement “the sum of the first n powers of two is $2^n - 1$.” We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \dots + 2^{k-1} = 2^k - 1. \quad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \quad (\text{via (1)}) \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

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New Stuff!

Variations on Induction: *Starting Later*

Induction Starting at 0

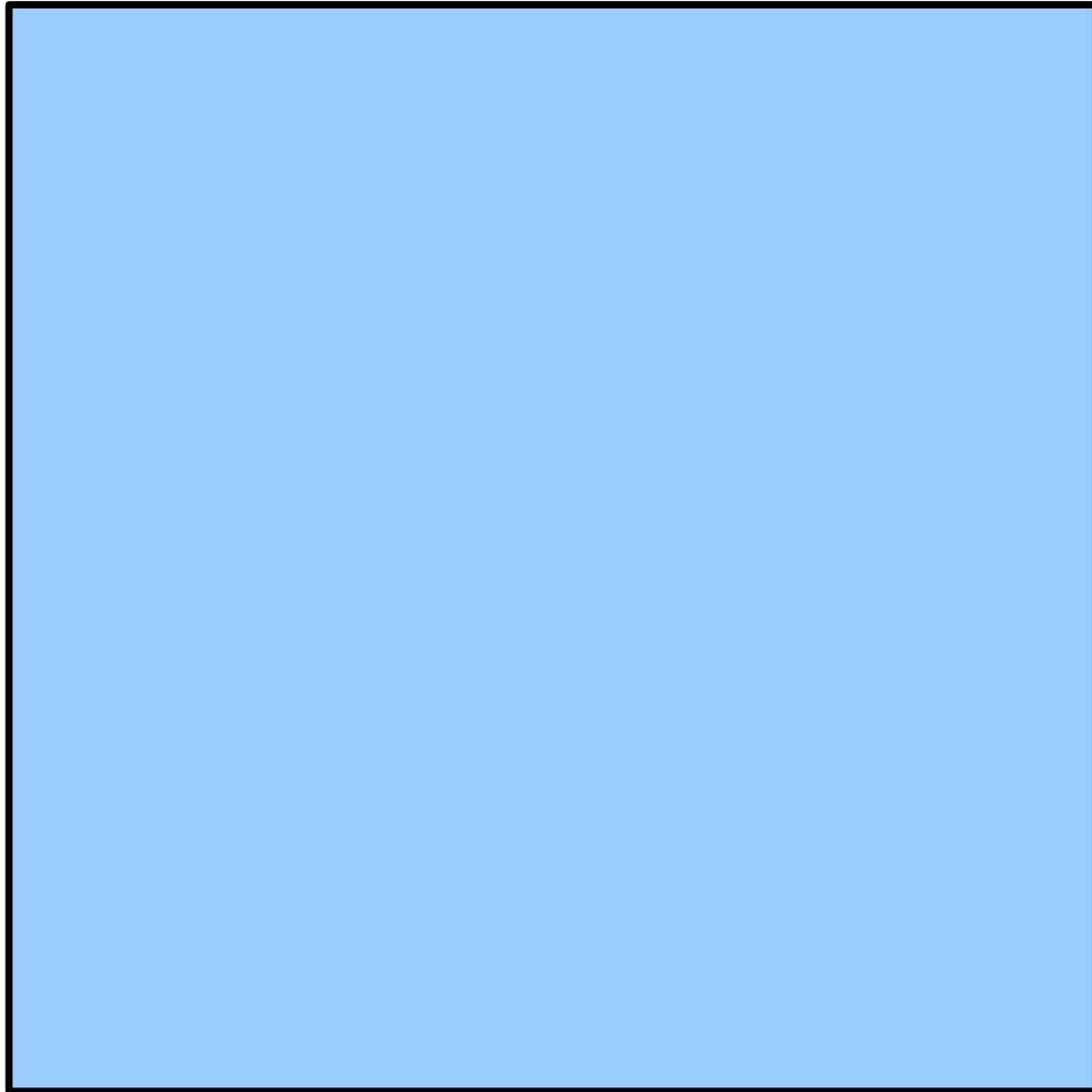
- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
 - Show that $P(0)$ is true.
 - Show that for any $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
 - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at m

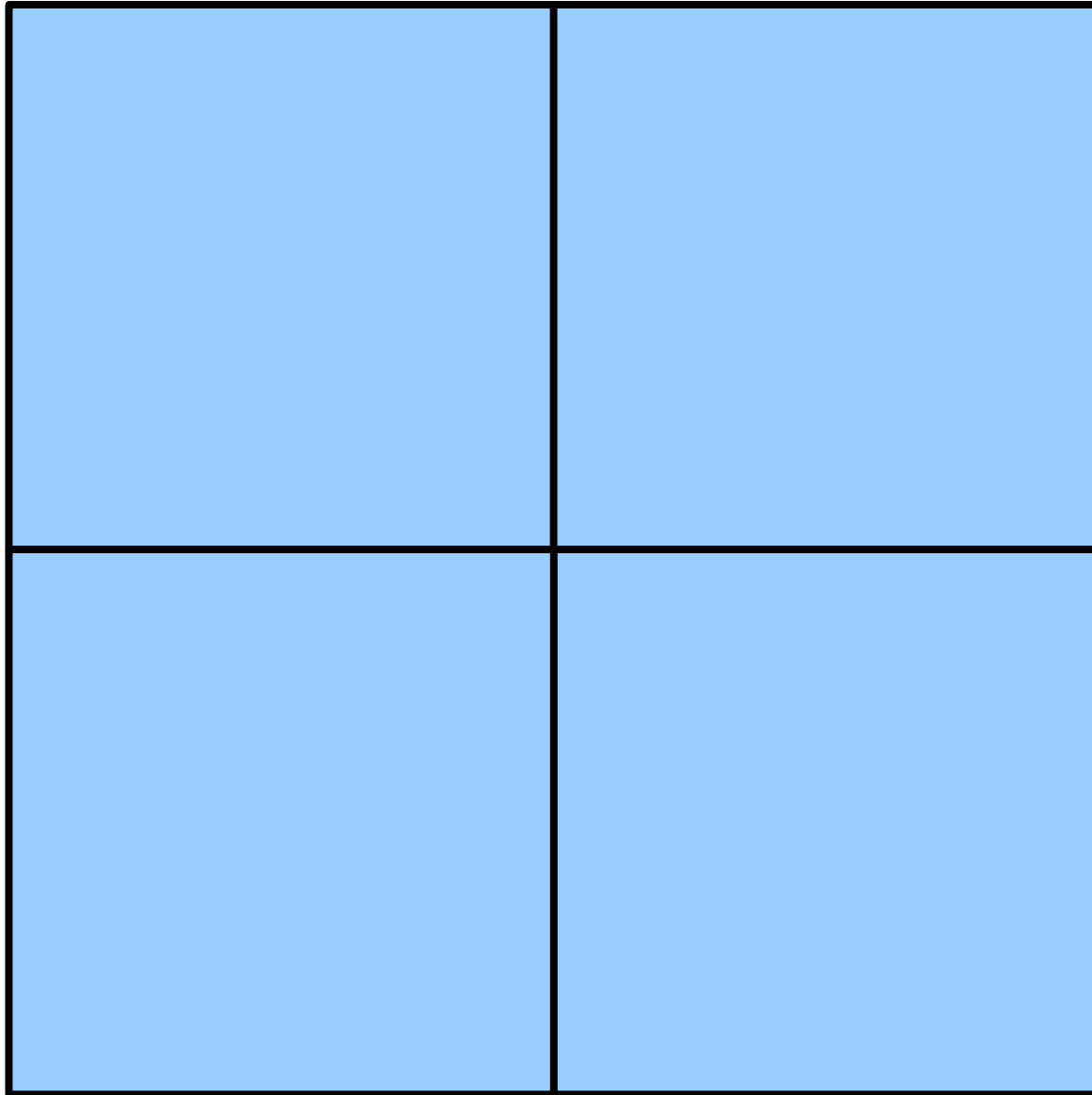
- To prove that $P(n)$ is true for all natural numbers greater than or equal to m :
 - Show that $P(m)$ is true.
 - Show that for any $k \geq m$, that if $P(k)$ is true, then $P(k+1)$ is true.
 - Conclude $P(n)$ holds for all natural numbers greater than or equal to m .

Variations on Induction: ***Bigger Steps***

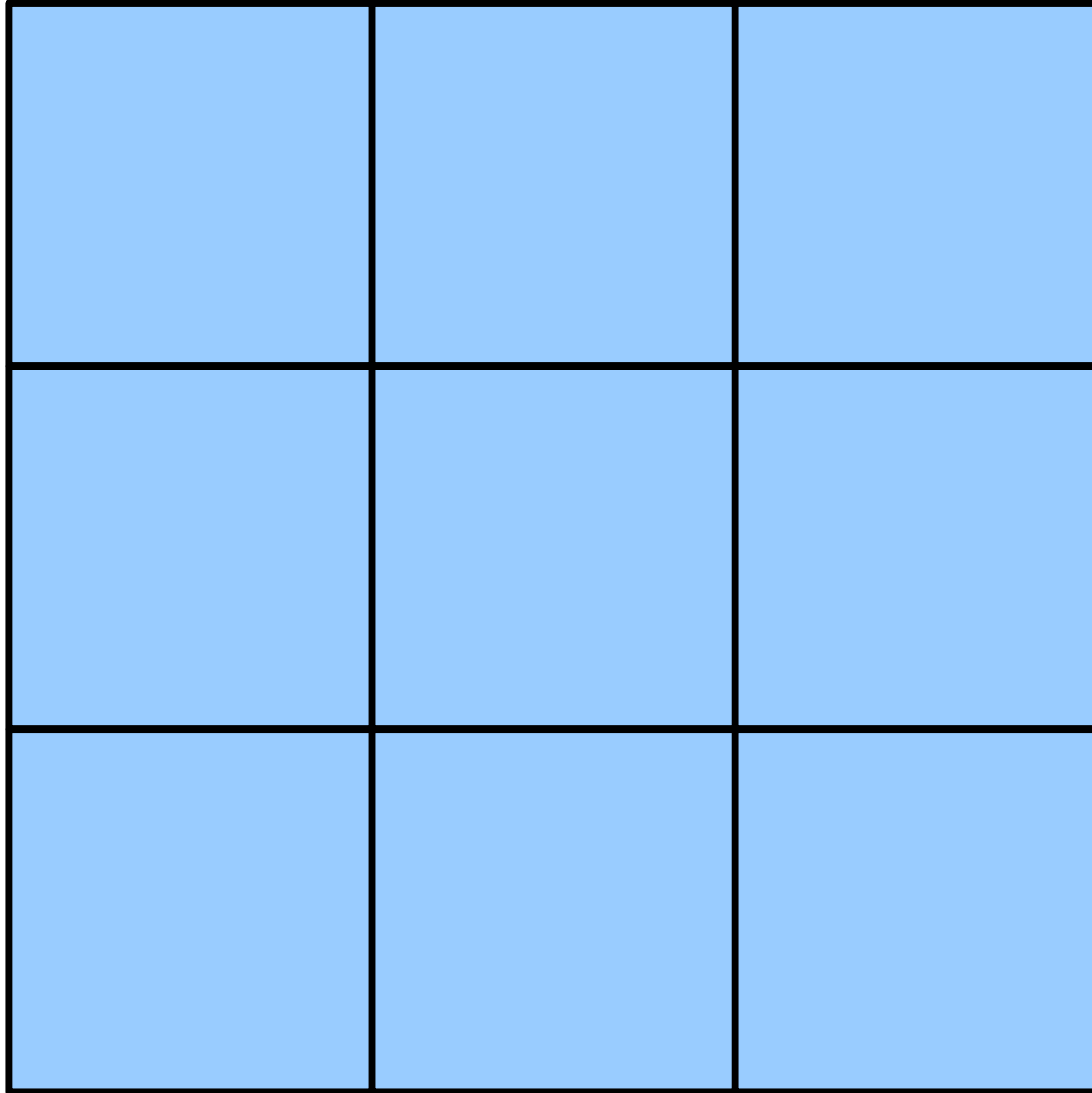
Subdividing a Square



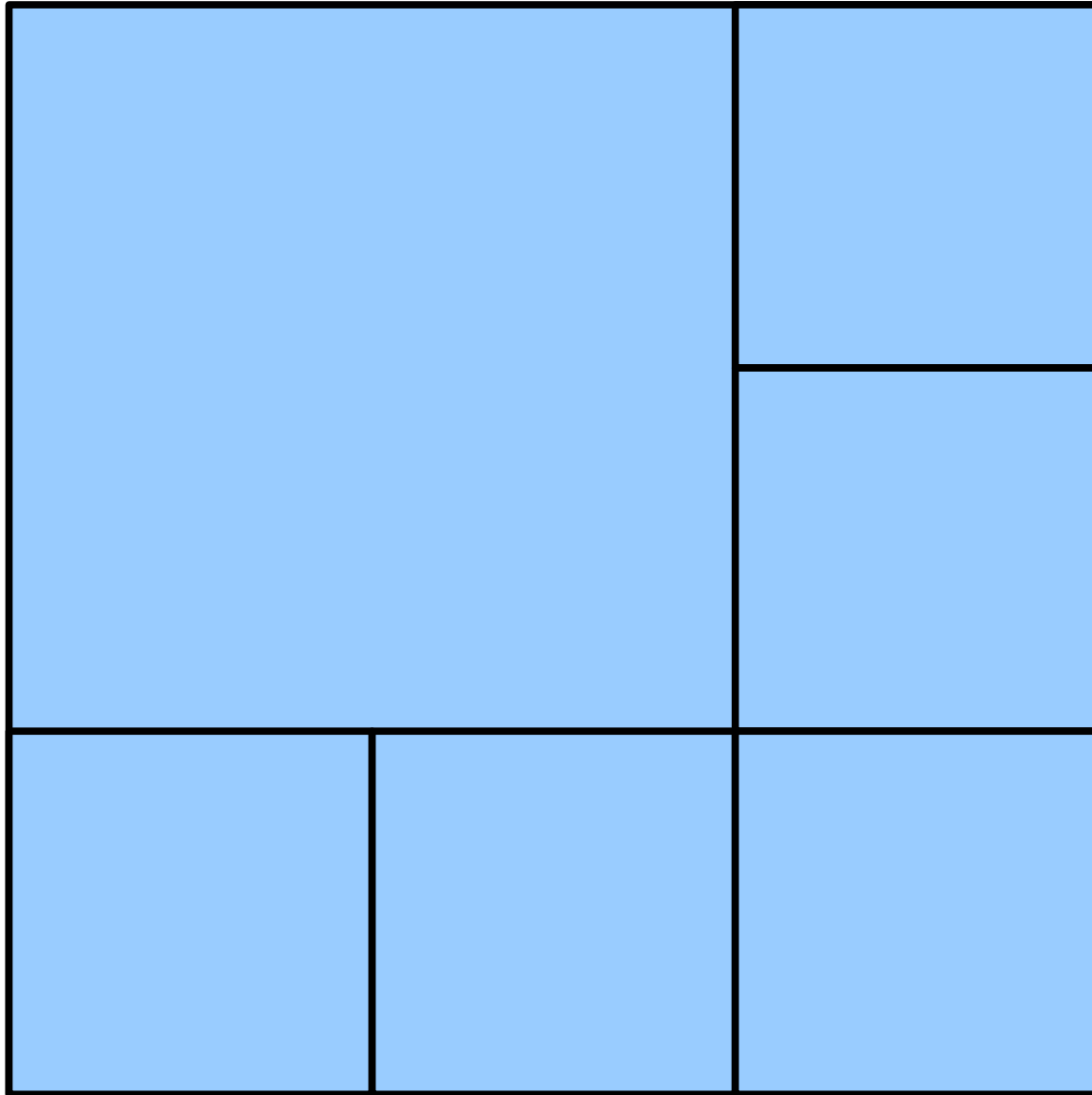
Subdividing a Square



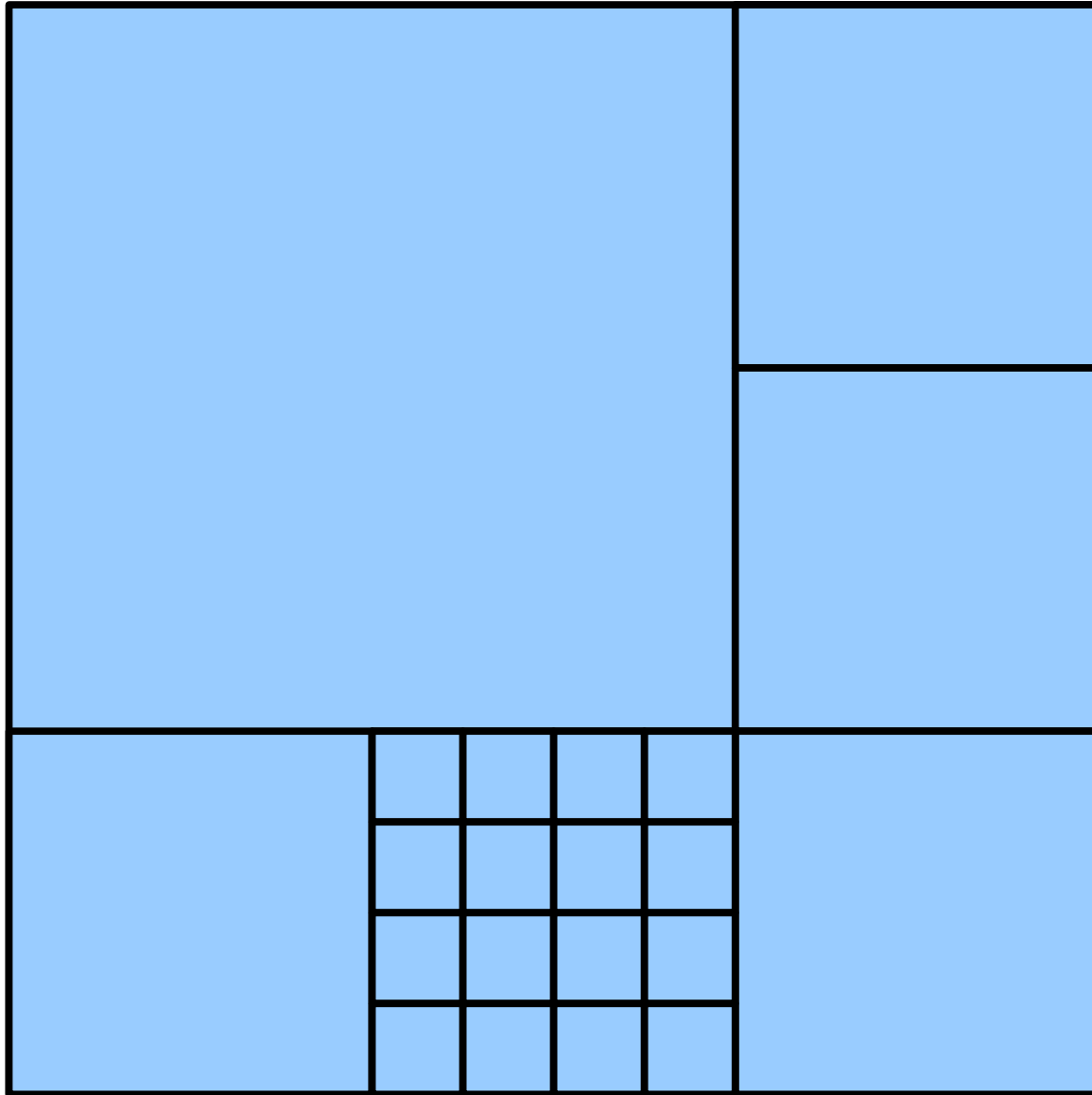
Subdividing a Square



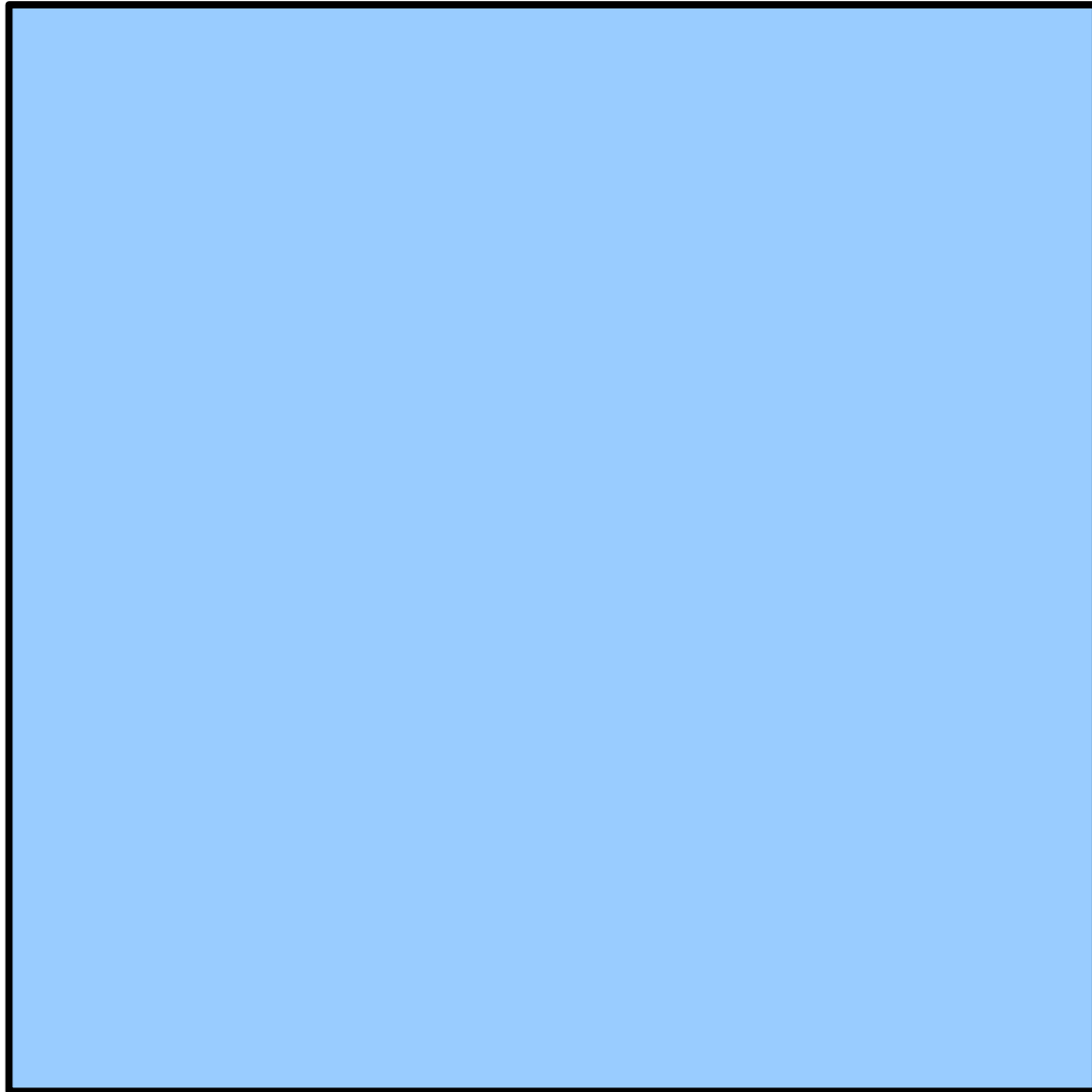
Subdividing a Square



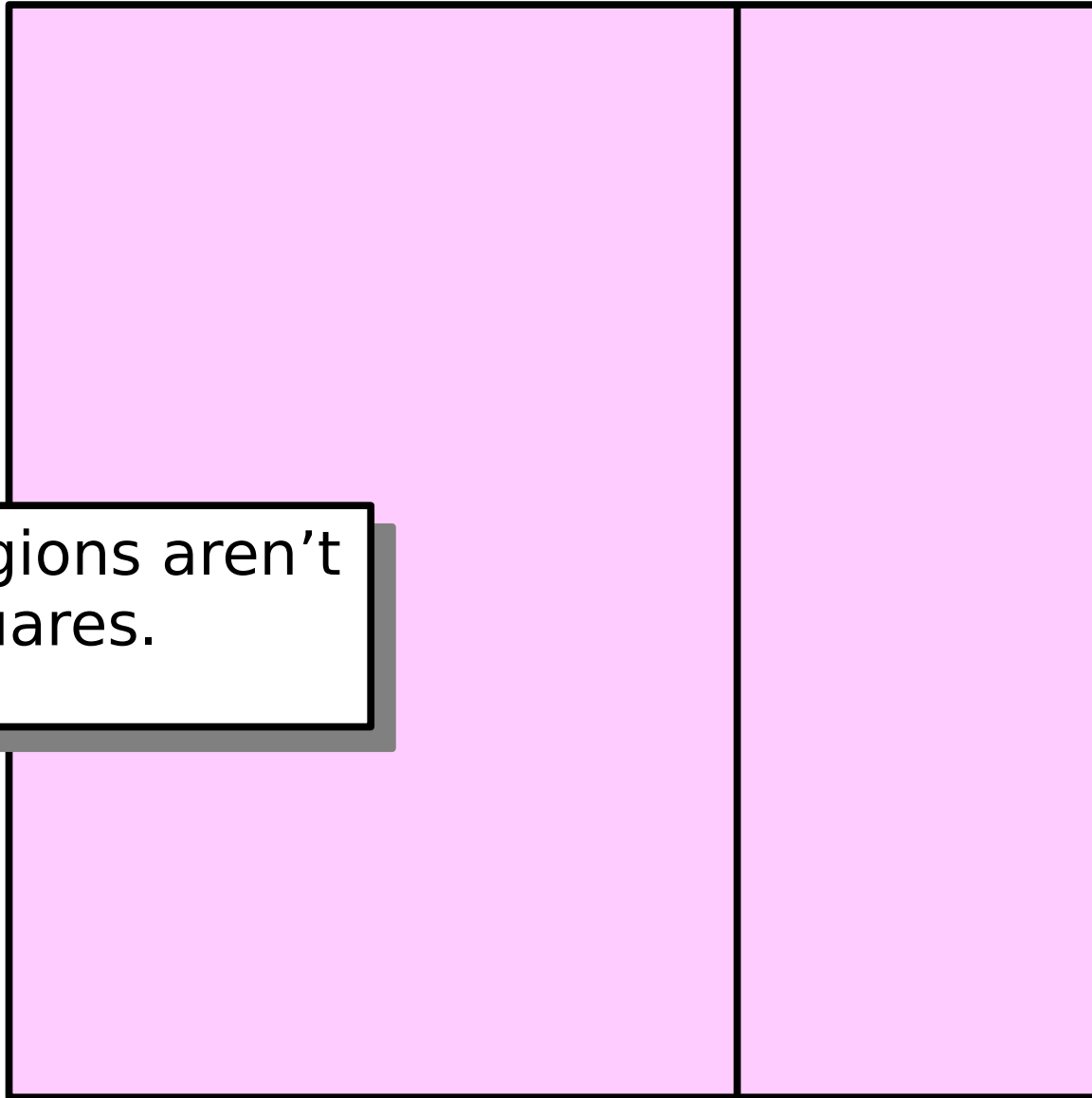
Subdividing a Square



Subdividing a Square

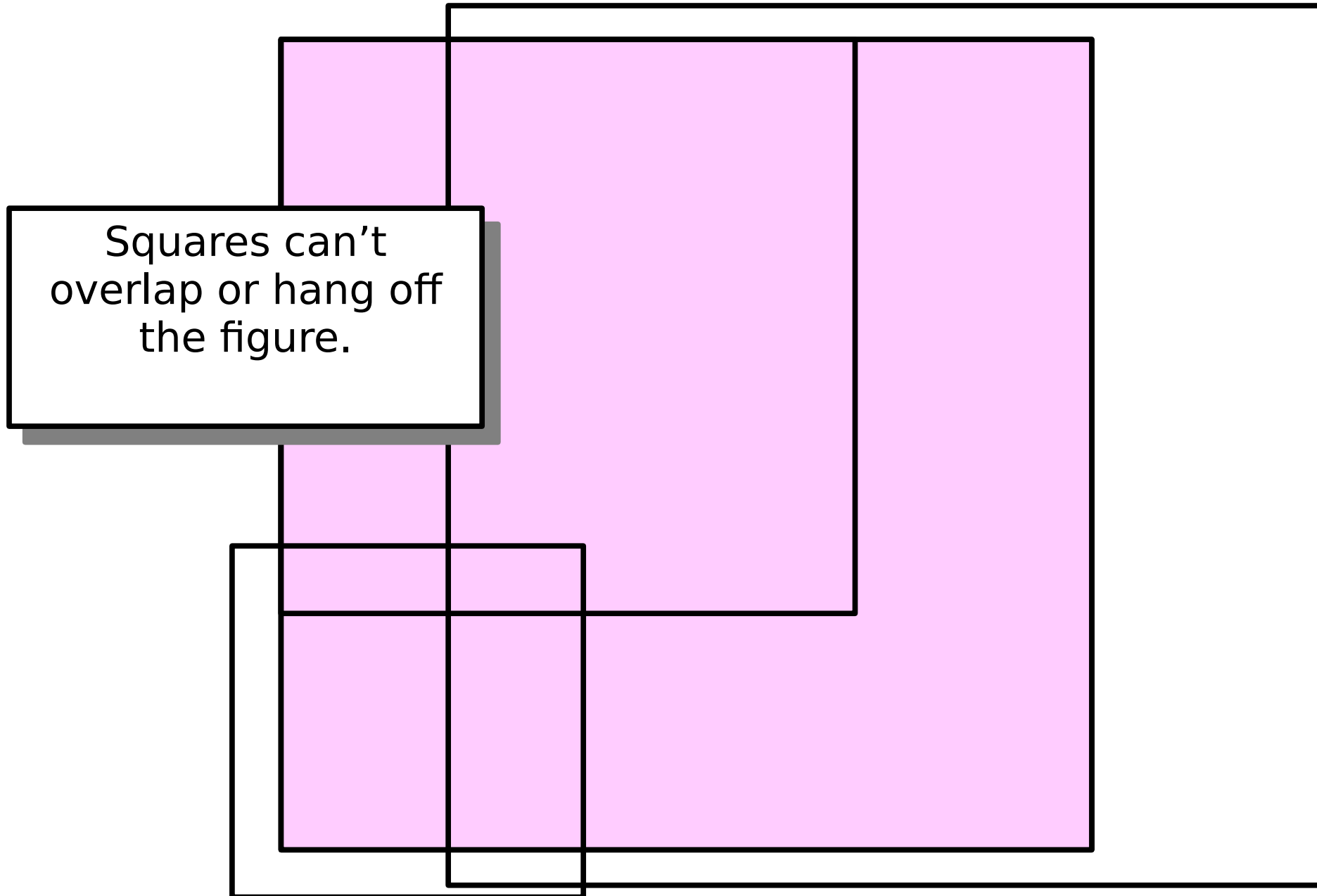


Subdividing a Square



These regions aren't squares.

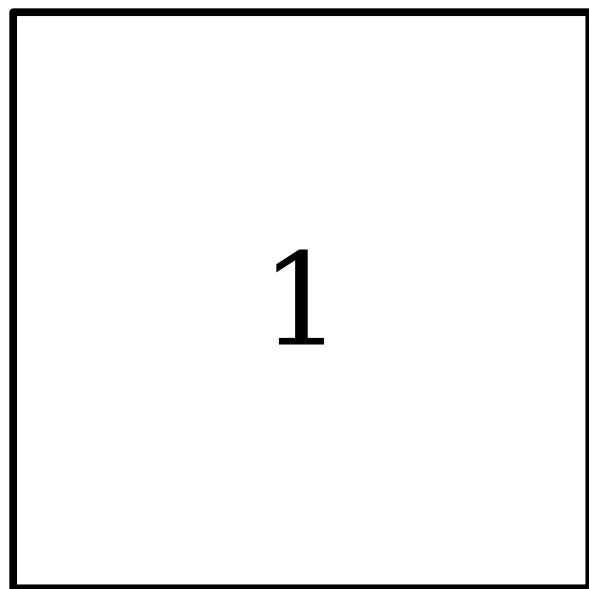
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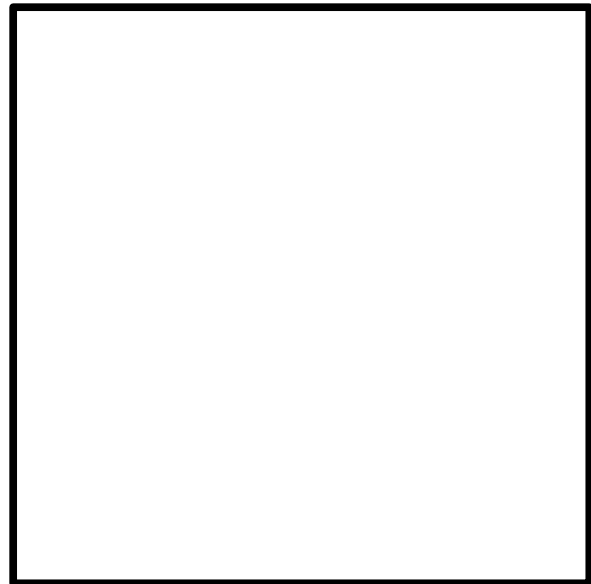
For what values of n can a square be subdivided into n squares?

1 2 3 4 5 6 7 8 9 10 11 12

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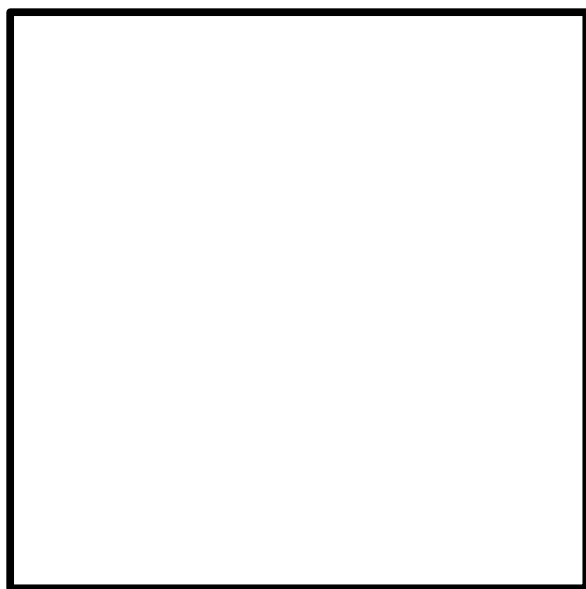


1 2 3 4 5 6 7 8 9 10 11 12



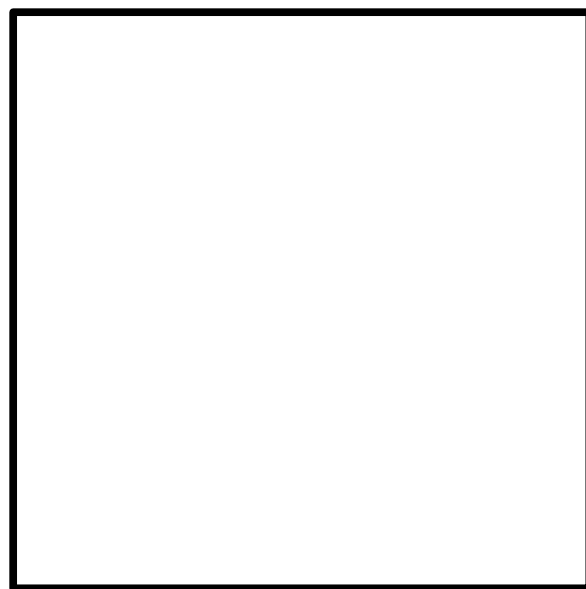
1 2 3 4 5 6 7 8 9 10 11 12

Each of the original corners needs to be covered by a corner of the new smaller squares.



1 2 3 4 5 6 7 8 9 10 11 12

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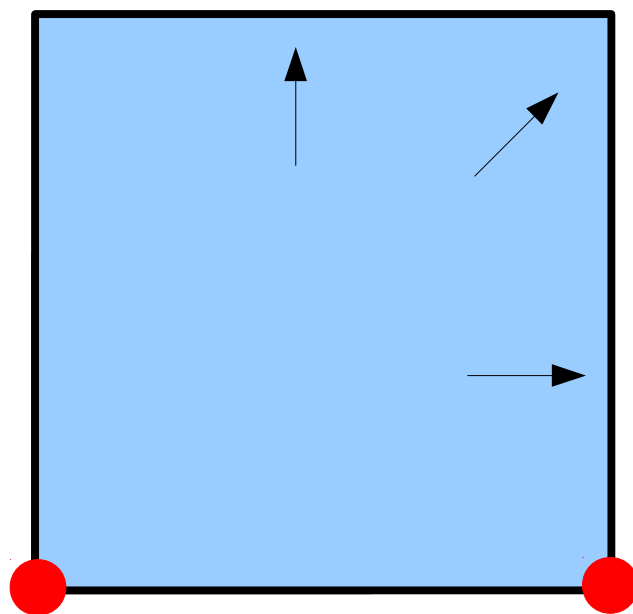


corners: 4

squares: <4

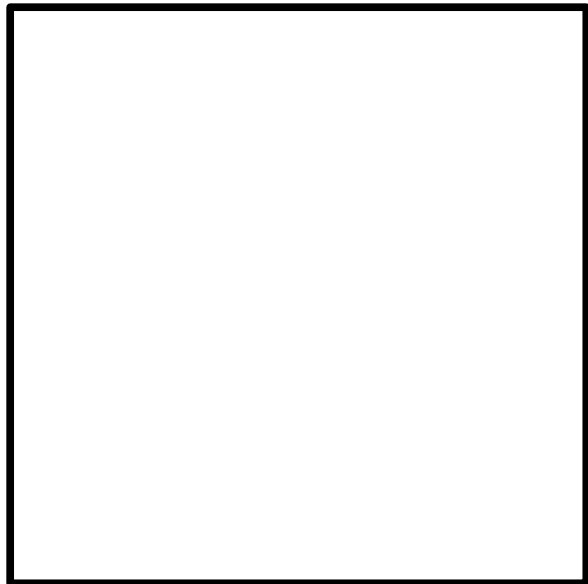
1 2 3 4 5 6 7 8 9 10 11 12

Each of the original corners needs to be covered by a corner of the new smaller squares.



By the pigeonhole principle, at least one smaller square needs to cover at least *two* of the original square's corners.

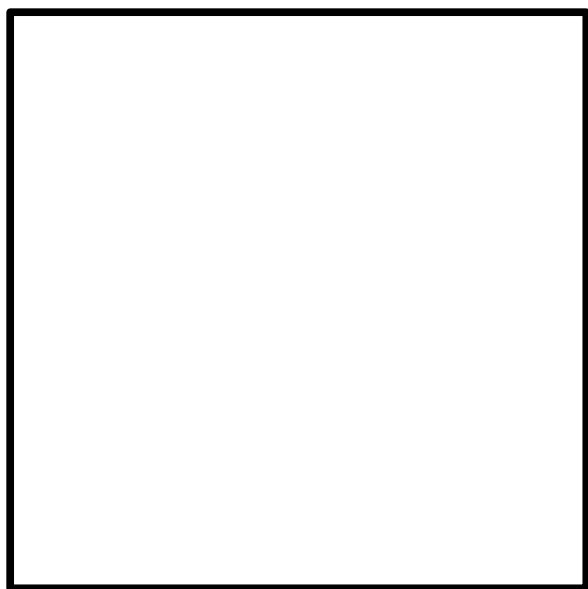
1 2 3 4 5 6 7 8 9 10 11 12



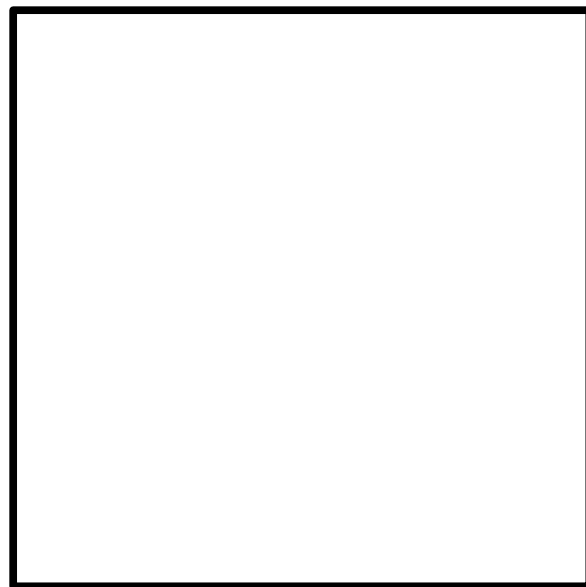
1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

1	2
4	3

1 2 3 4 5 6 7 8 9 10 11 12



1 2 3 4 5 6 7 8 9 10 11 12

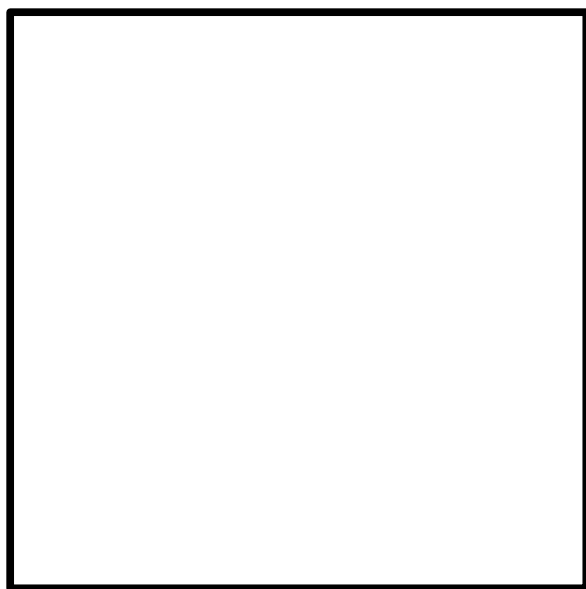


corners: 4

squares: 5

1 2 3 4 5 6 7 8 9 10 11 12

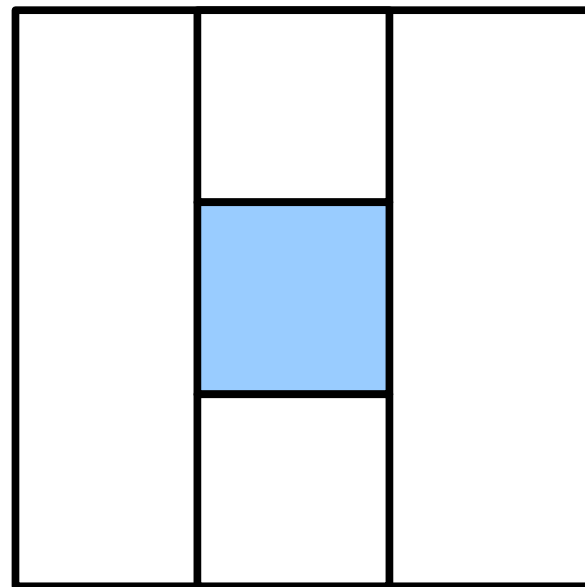
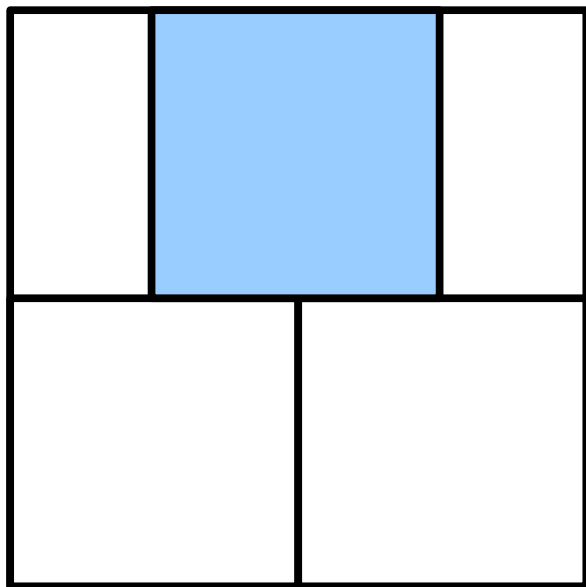
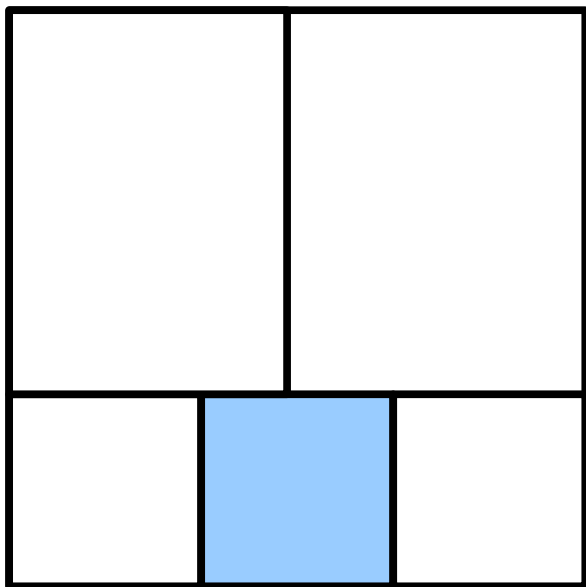
At least one square
cannot be covering
any of the original
corners



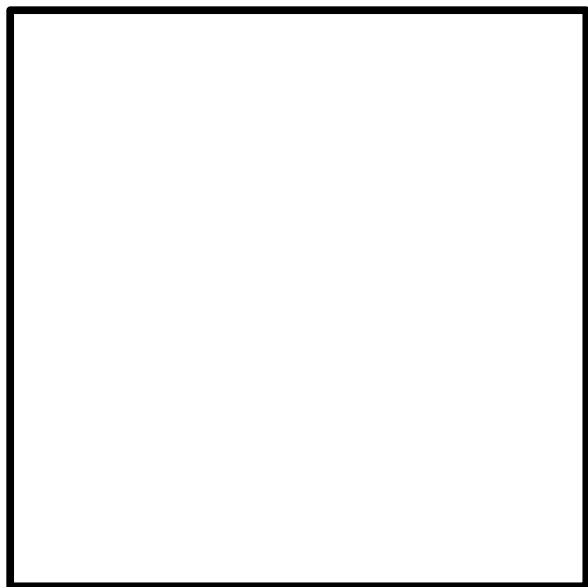
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1 2 3 4 5 6 7 8 9 10 11 12



1 2 3 4 5 6 7 8 9 10 11 12



1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1		2
		3
6	5	4

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

5	6	1
4	7	
3		2

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1			
2	8		
3			
4	5	6	7

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	2	3
8	9	4
7	6	5

1 2 3 4 5 6 7 8 9 10 11 12

1	2	3	
8	9	3	
7		10	4
		6	5

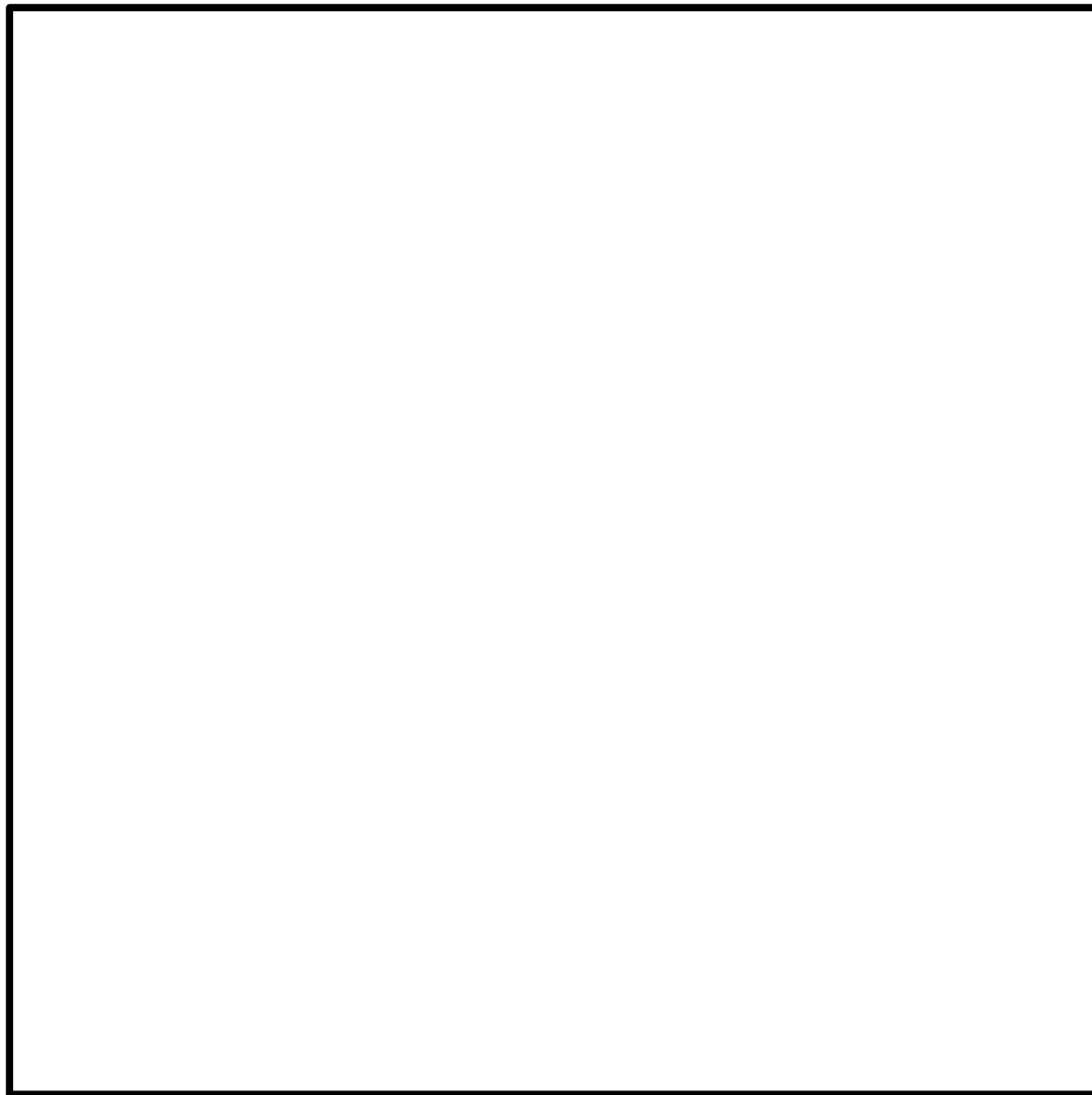
1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	10		9
2	11		8
3	5	6	7
4			

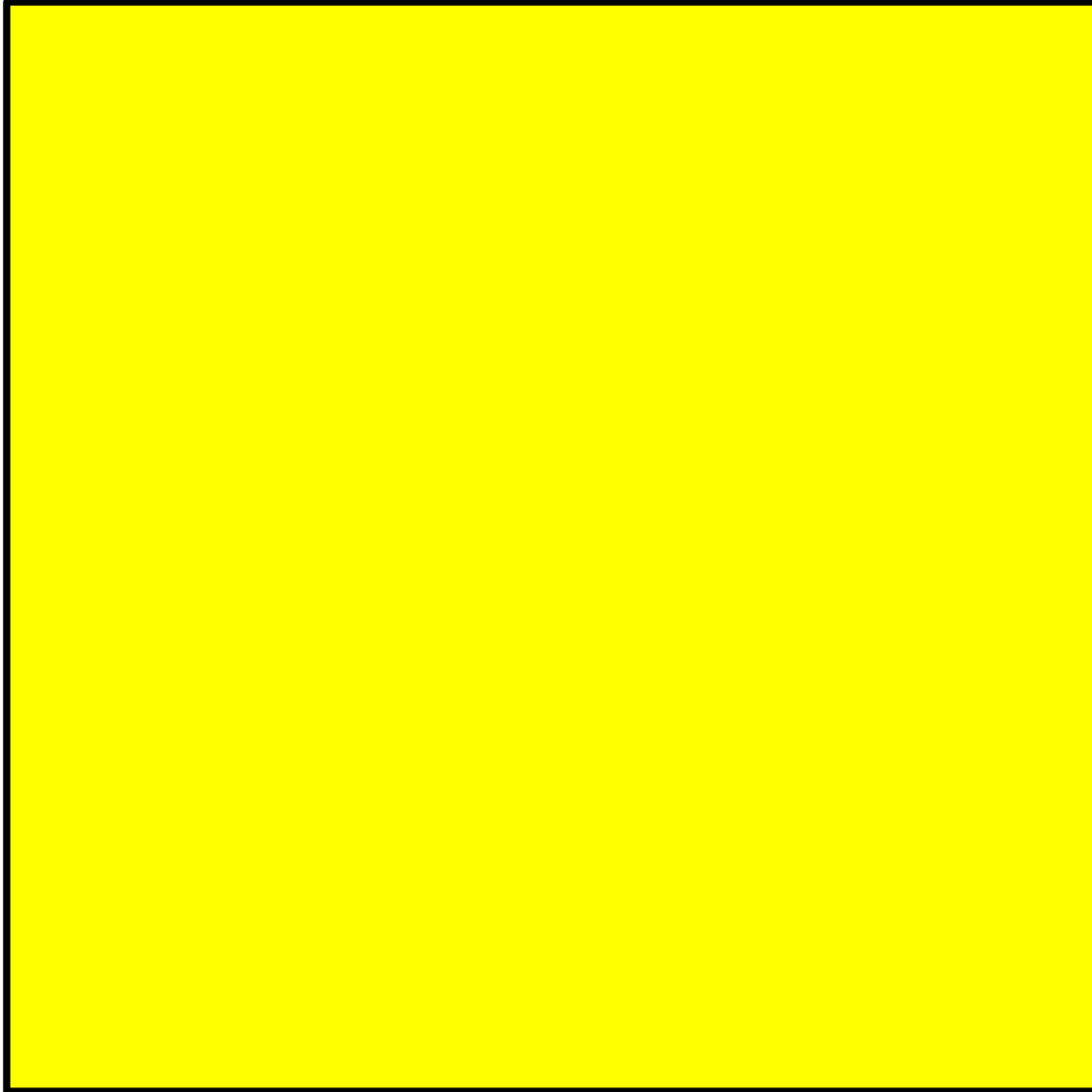
1 2 3 4 5 6 7 8 9 10 11 12

1	2	3
8	9 10	4
7	12 11	5
6	6	5

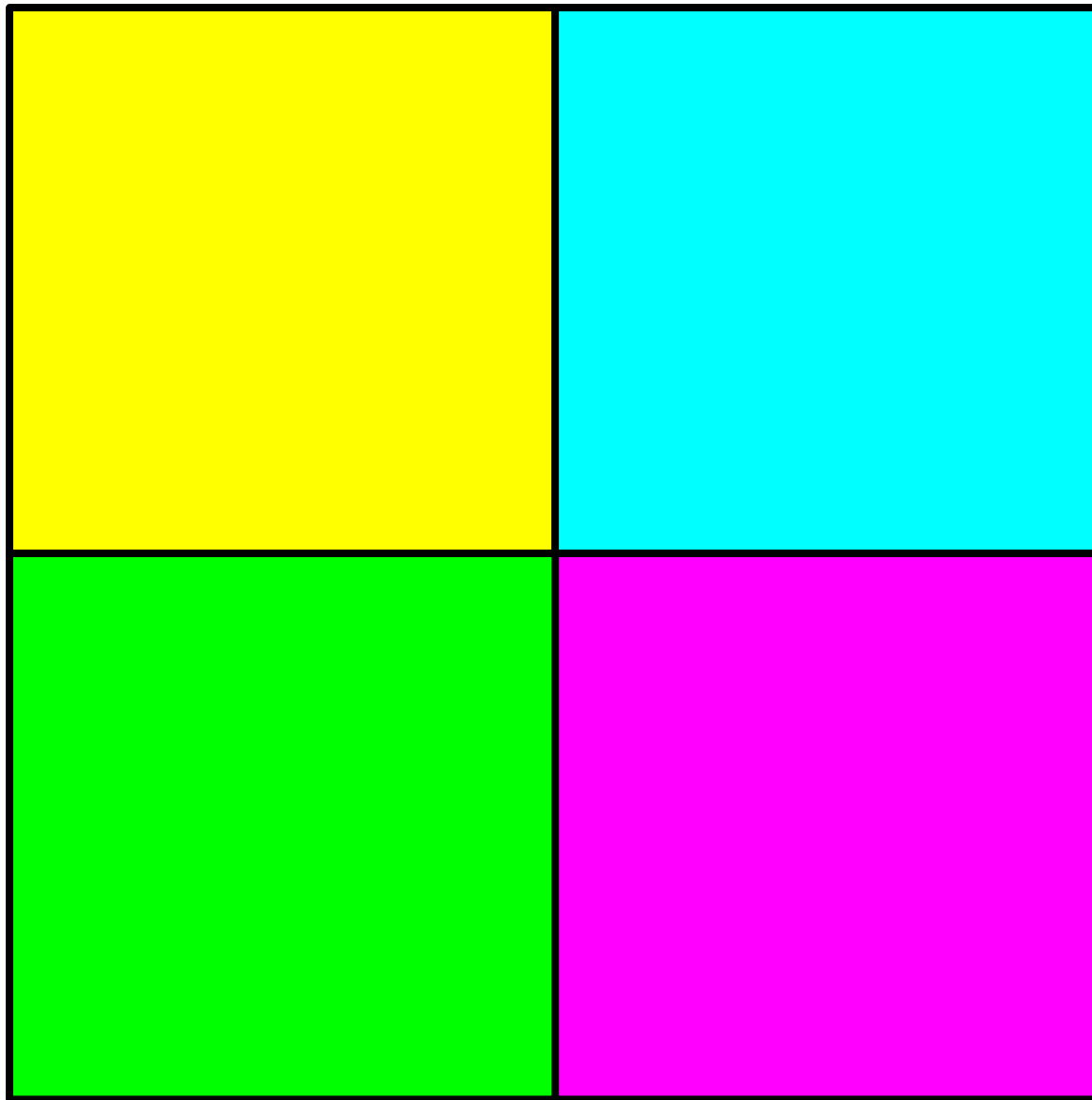
An Insight



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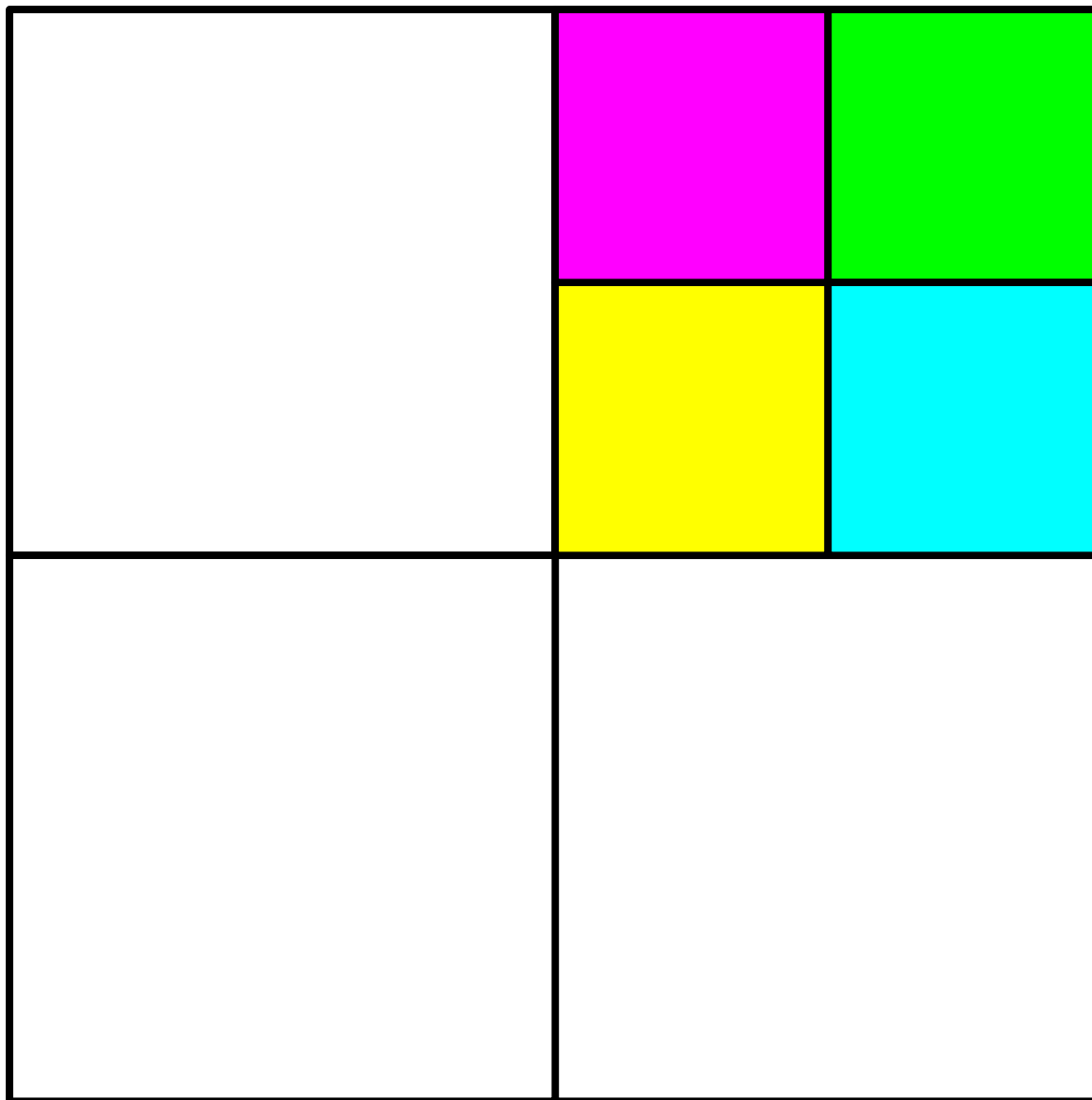
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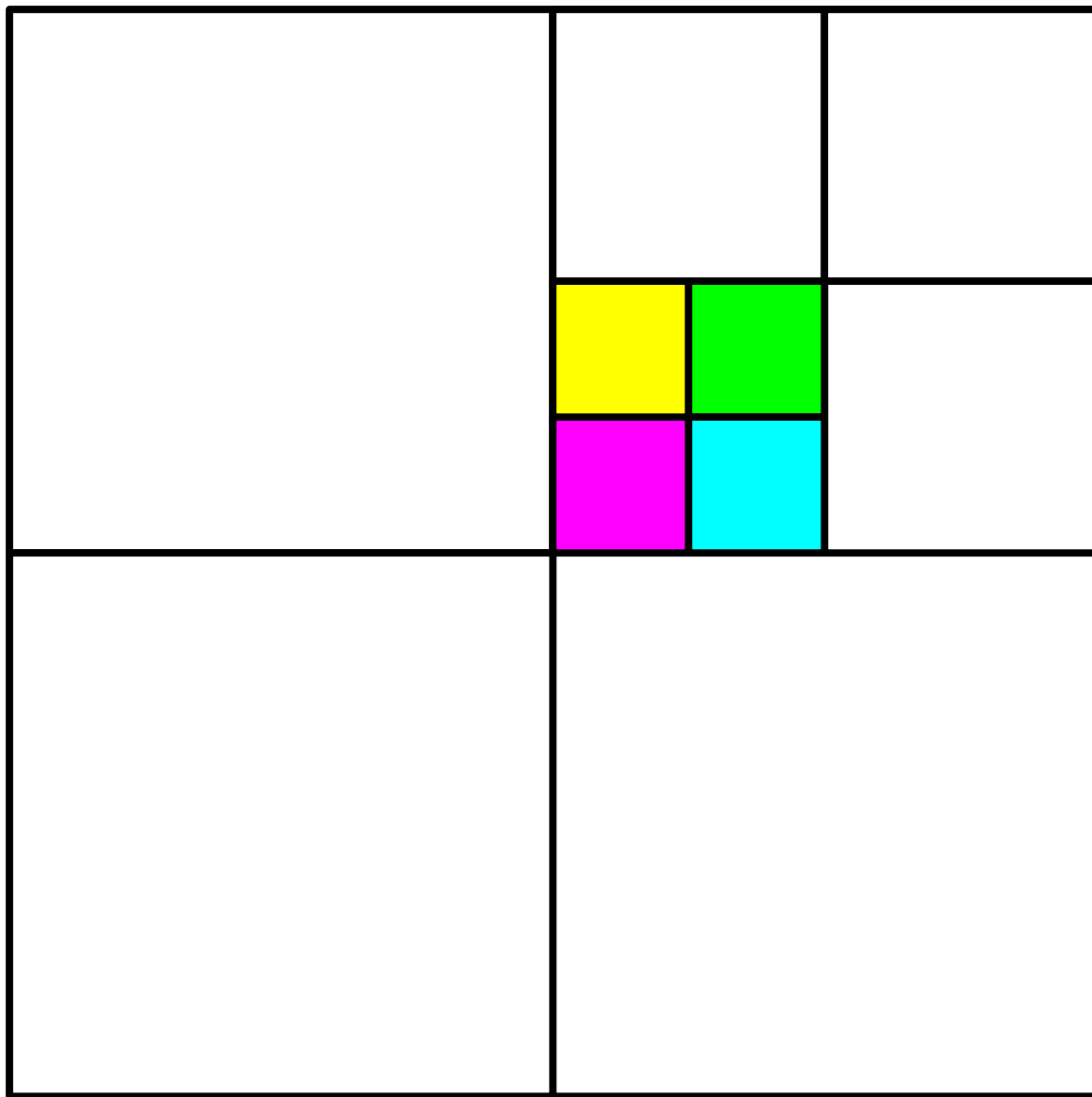
An Insight



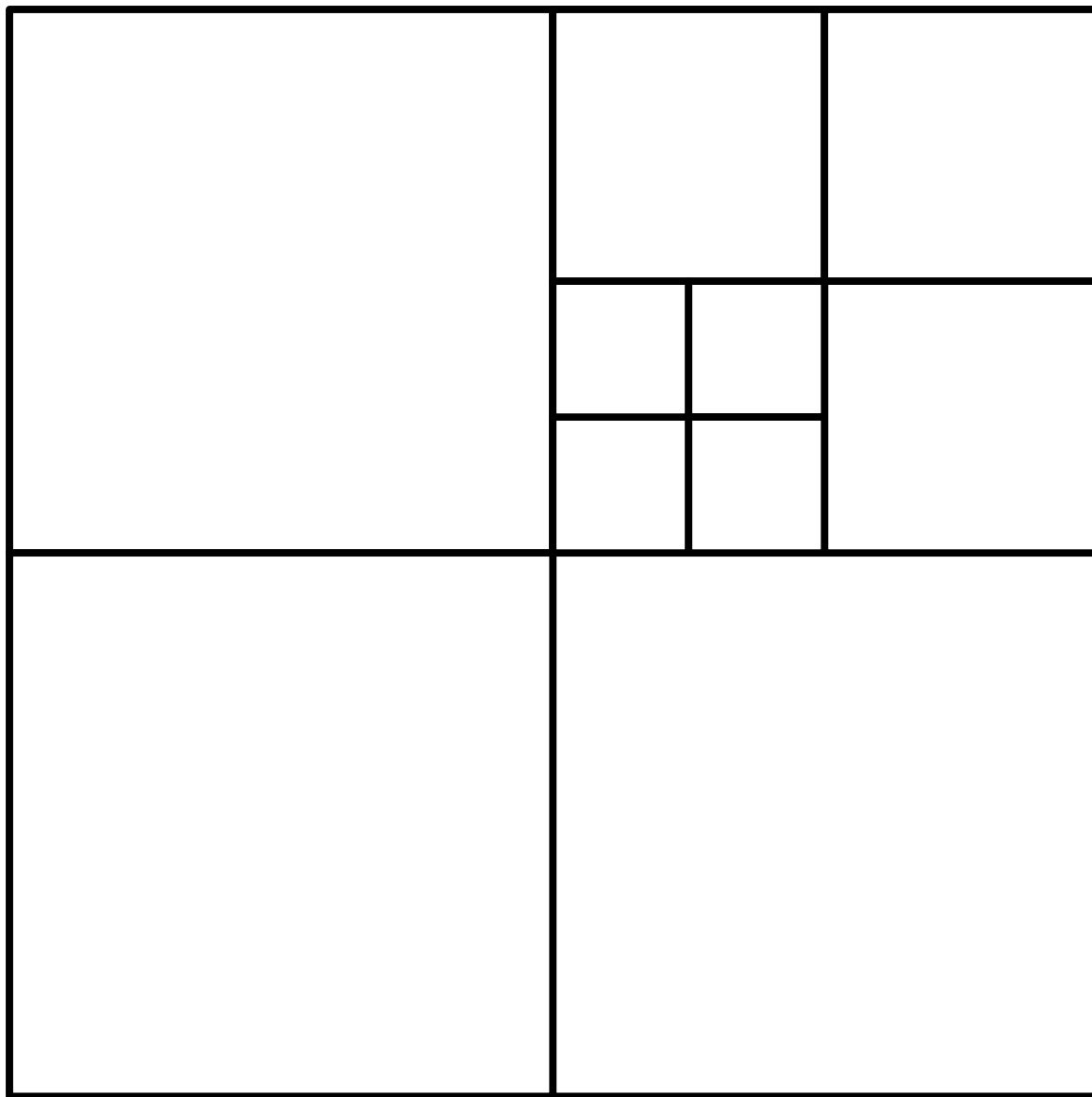
An Insight

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An Insight



An Insight



An Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
 - For multiples of three, start with 6 and keep adding three squares until n is reached.
 - For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
 - For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

Theorem: For any $n \geq 6$, there is a way to subdivide a square into n smaller squares.

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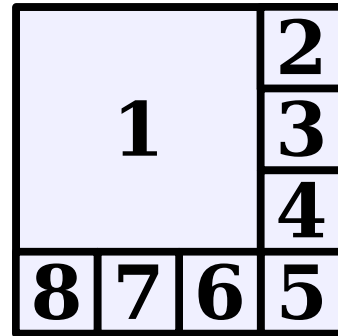
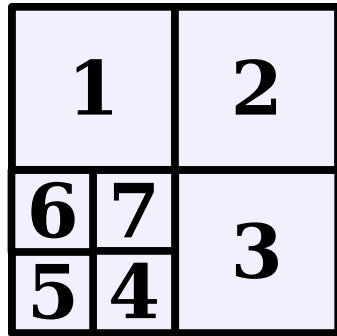
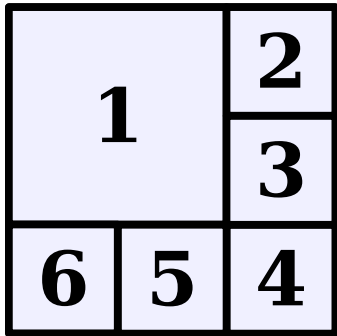
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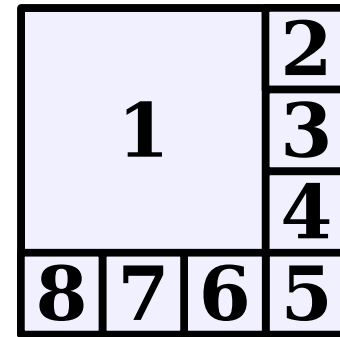
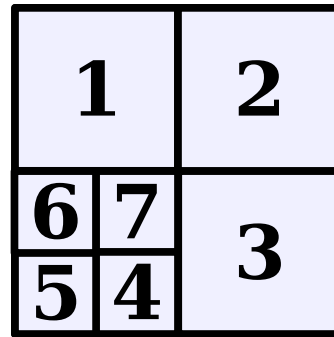
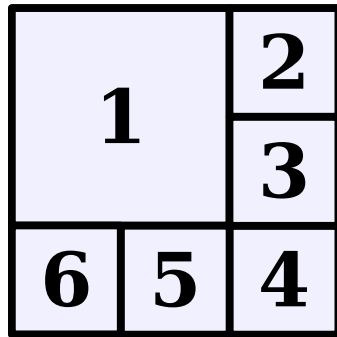
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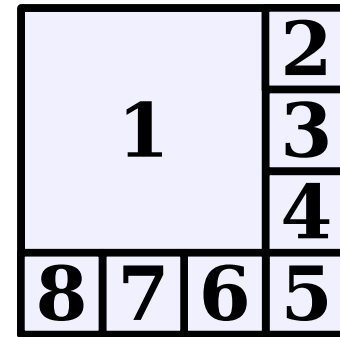
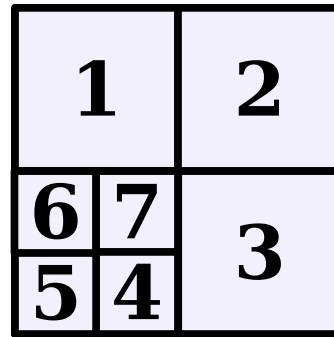
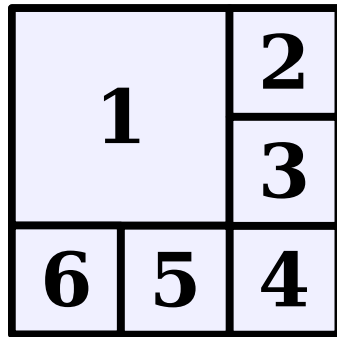


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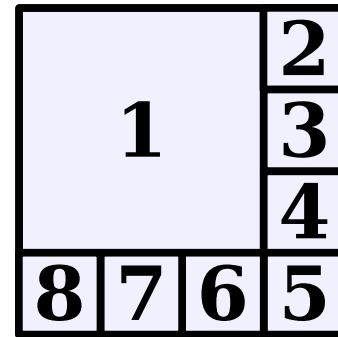
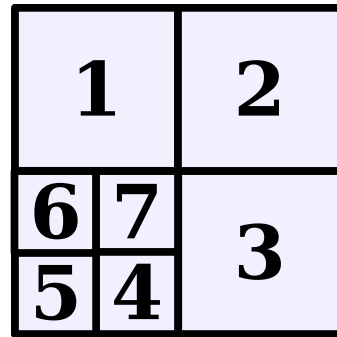
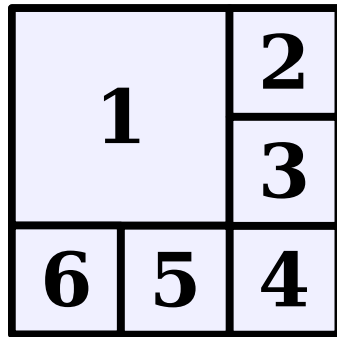


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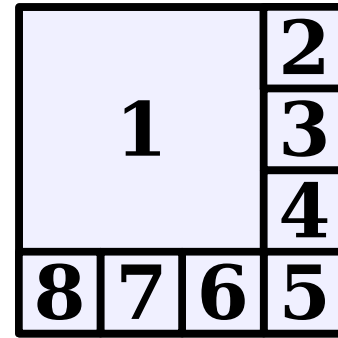
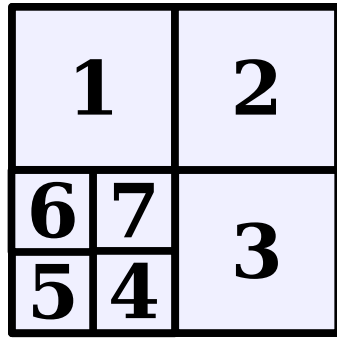
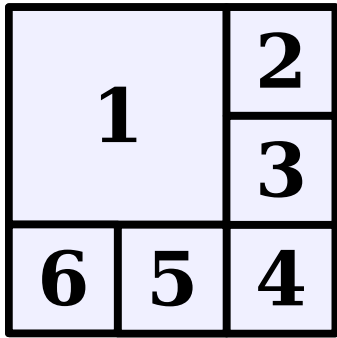


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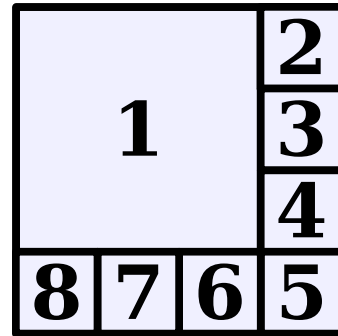
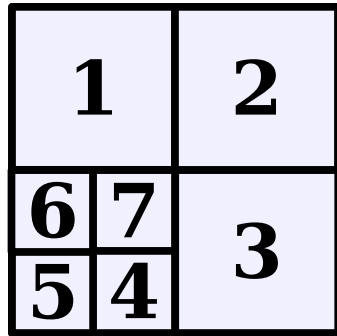
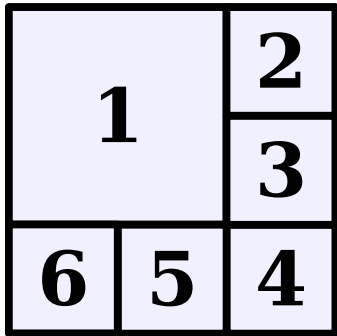


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Proof: Let $P(n)$ be the statement “there is a way to subdivide a square into n smaller squares.” We will prove by induction that $P(n)$ holds for all $n \geq 6$, from which the theorem follows.

As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares. This is shown here:

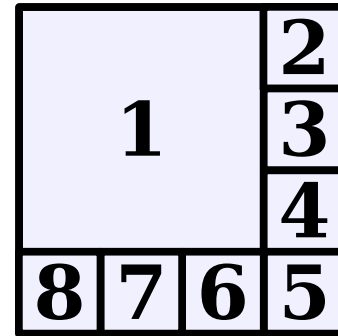
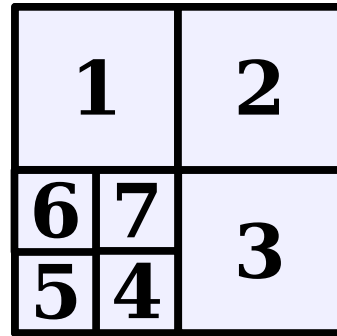
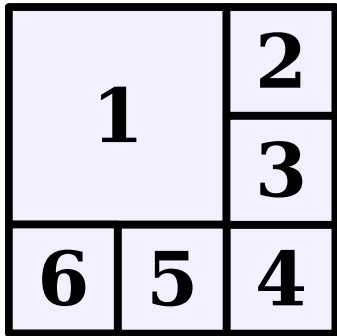


For the inductive step, assume that for some arbitrary $k \geq 6$ that $P(k)$ is true and that there is a way to subdivide a square into k squares. We prove $P(k+3)$, that there is a way to subdivide a square into $k+3$ squares. To see this, start by obtaining (via the inductive hypothesis) a subdivision of a square into k squares. Then, choose any of the squares and split it into four equal squares. This removes one of the k squares and adds four more, so there will be a net total of $k+3$ squares.

Theorem: For any $n \geq 6$, there is a way to subdivide a square into n smaller squares.

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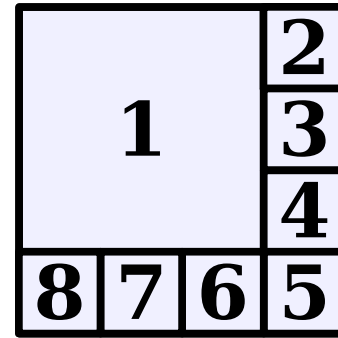
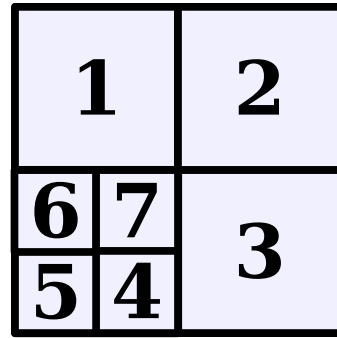
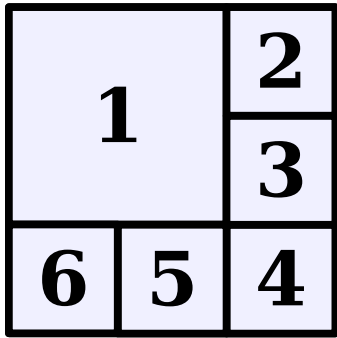


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Generalizing Induction

- When doing a proof by induction,
 - feel free to use multiple base cases, and
 - feel free to take steps of sizes other than one.
- If you do, make sure that...
 - ... you actually need all your base cases. Avoid redundant base cases that are already covered by a mix of other base cases and your inductive step.
 - ... you cover all the numbers you need to cover. Trace out your reasoning and make sure all the numbers you need to cover really are covered.
- As with a proof by cases, you don't need to separately prove you've covered all the options. We trust you. 😊

More on Square Subdivisions

- There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.
- In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.
- Good starting resource: this Numberphile video on [*Squaring the Square*](#).

Complete Induction

Let P be some predicate. The ***principle of complete induction*** states that if

If it starts true...

$P(0)$ is true

...and it stays true...

and

for all $k \in \mathbb{N}$, if $P(0)$, ..., and $P(k)$ are true, then $P(k+1)$ is true

then

$\forall n \in \mathbb{N}. P(n)$

...then it's always true.

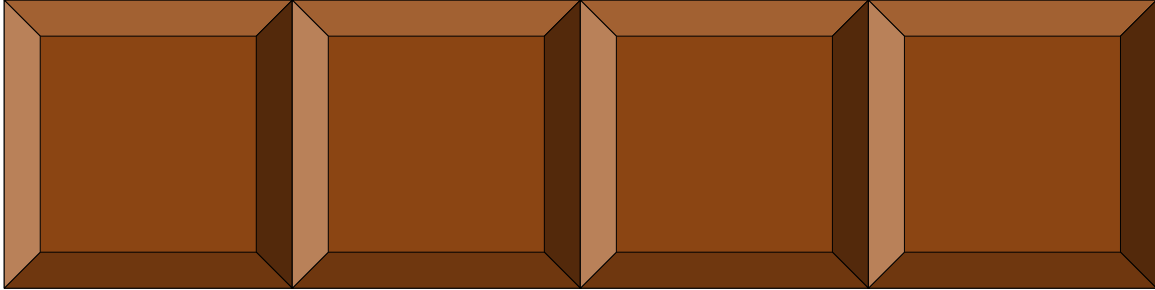
Mathematical Induction

- You can write proofs using the principle of mathematical induction as follows:
 - Define some predicate $P(n)$ to prove by induction on n .
 - Choose and prove a base case (probably, but not always, $P(0)$).
 - Pick an arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true.
 - Prove $P(k+1)$.
 - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

Complete Induction

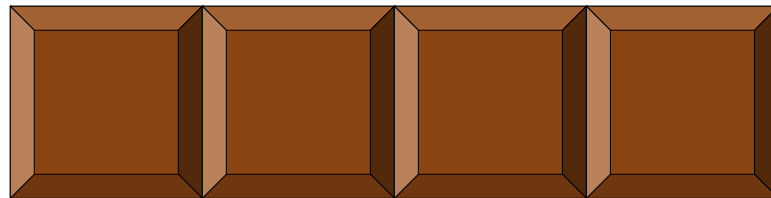
- You can write proofs using the principle of **complete** induction as follows:
 - Define some predicate $P(n)$ to prove by induction on n .
 - Choose and prove a base case (probably, but not always, $P(0)$).
 - Pick an arbitrary $k \in \mathbb{N}$ and assume that **$P(0), P(1), P(2), \dots,$ and $P(k)$** are all true.
 - Prove $P(k+1)$.
 - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

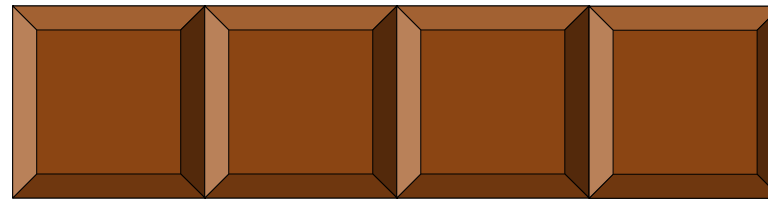
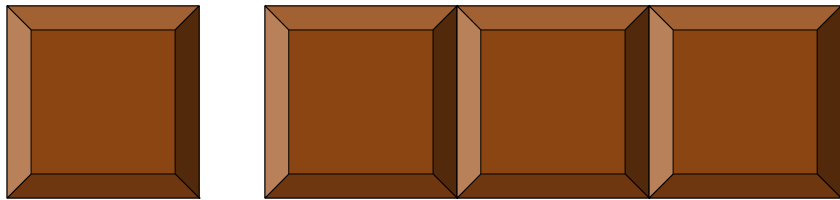
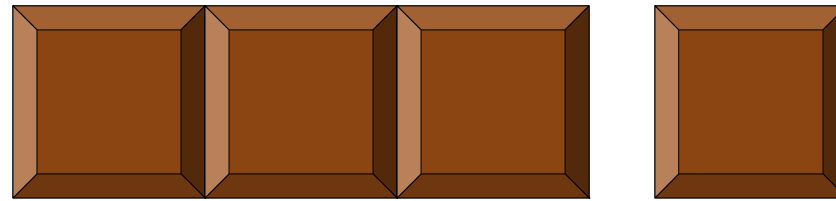
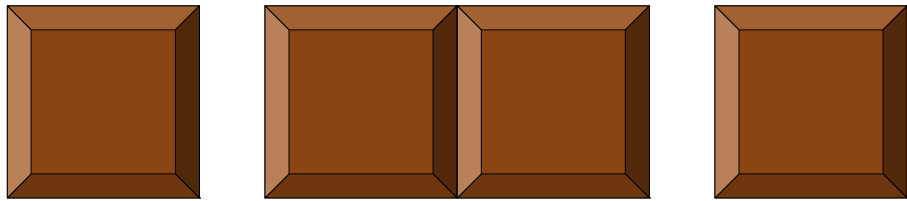
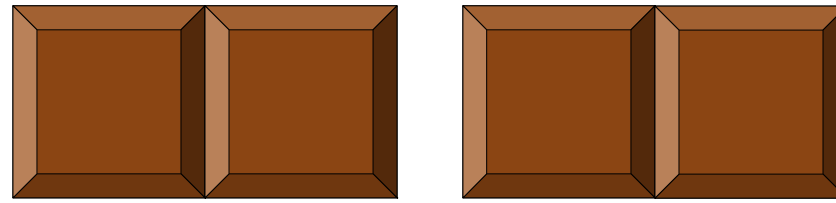
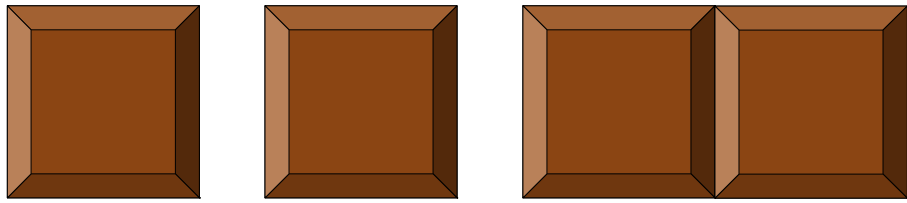
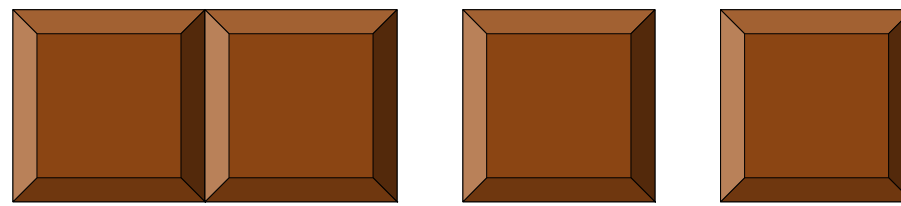
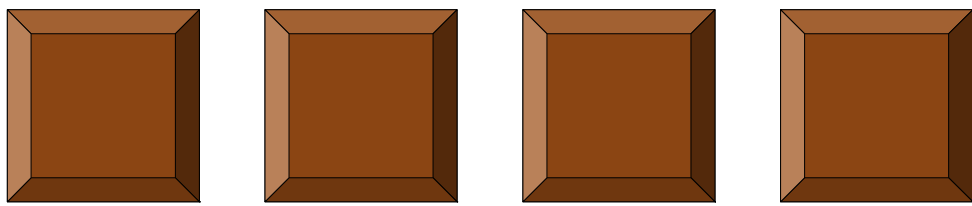
An Example: *Eating a Chocolate Bar*



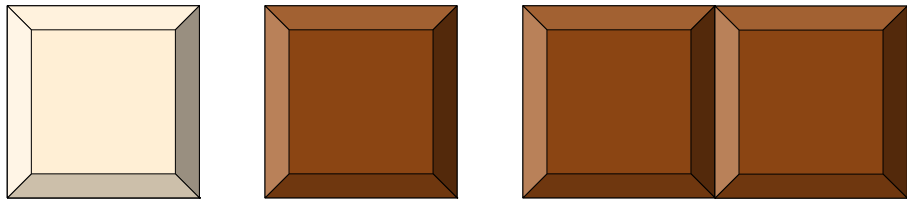
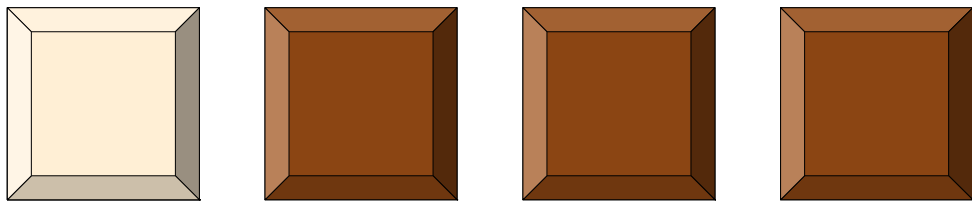
Eating a Chocolate Bar

- You have a $1 \times n$ chocolate bar subdivided into 1×1 squares.
- You eat the chocolate bar from left to right by breaking off one or more squares and eating them in one (possibly enormous) bite.
- How many ways can you eat a...
 - 1×1 chocolate bar?
 - 1×2 chocolate bar?
 - 1×3 chocolate bar?
 - 1×4 chocolate bar?

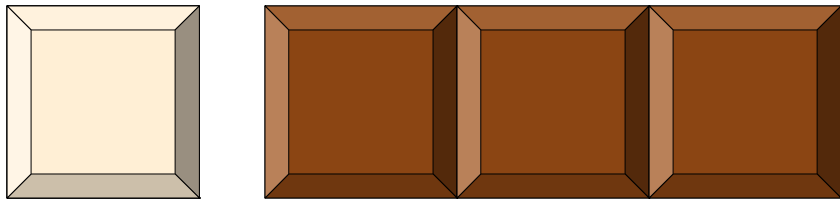
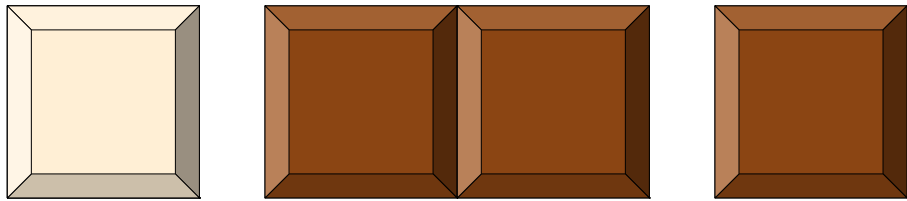




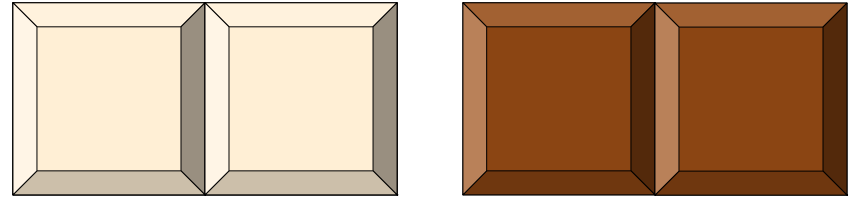
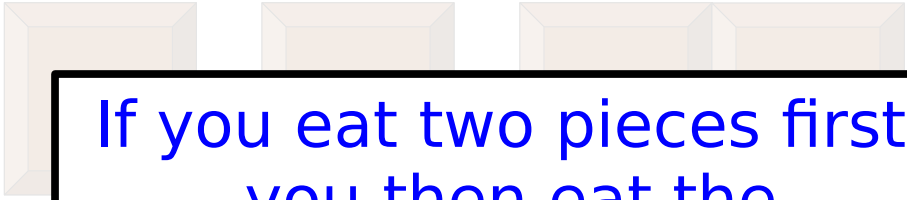
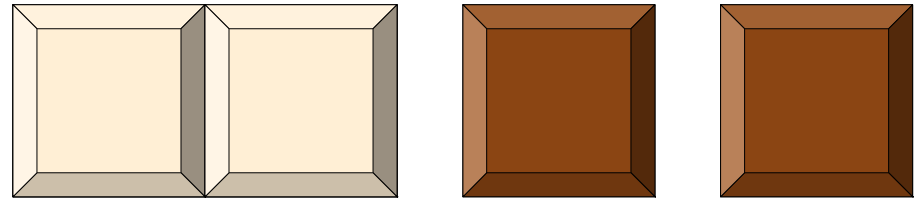
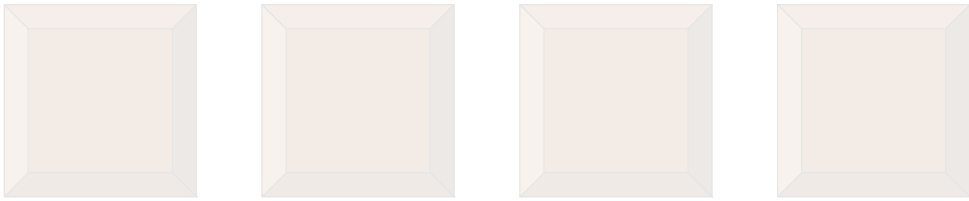
There are eight ways to eat a 1×4 chocolate bar.



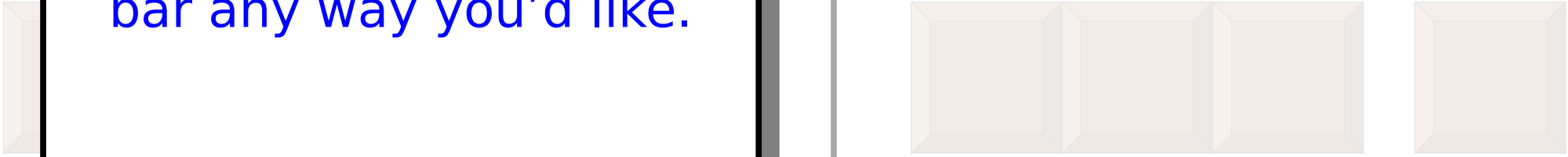
If you eat one piece first,
you then eat the
remaining 1×3 chocolate
bar any way you'd like.



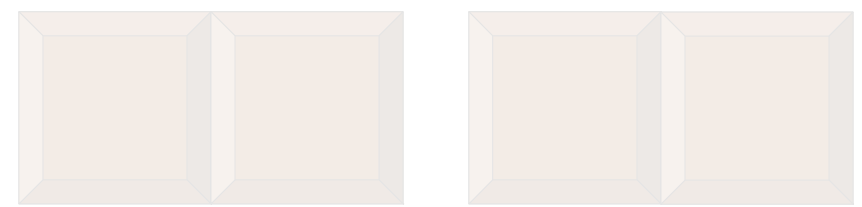
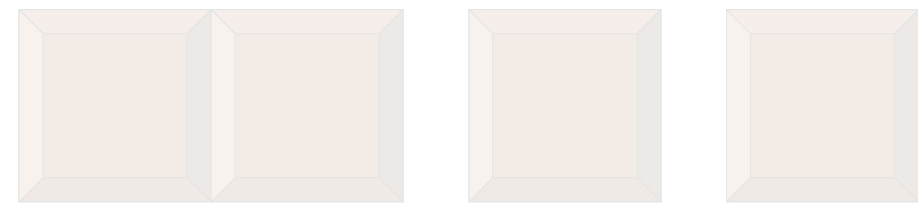
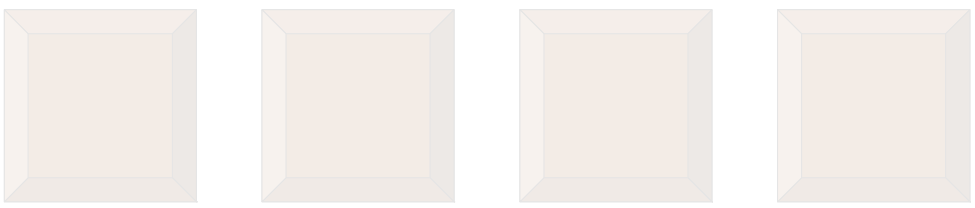
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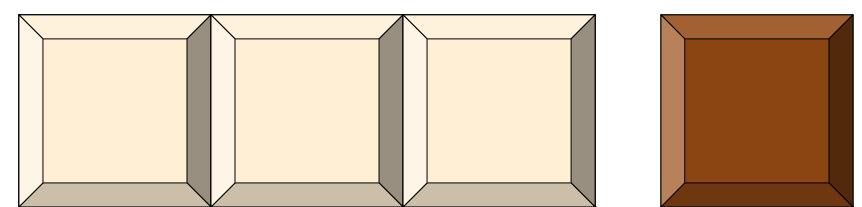
If you eat two pieces first,
you then eat the
remaining 1×2 chocolate
bar any way you'd like.



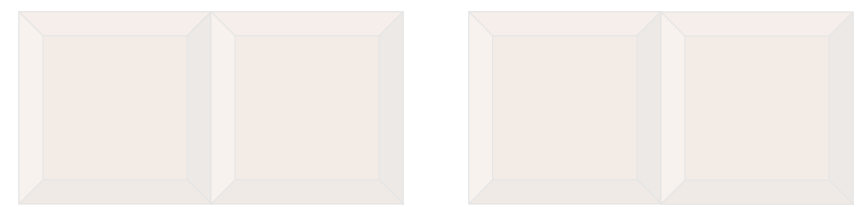
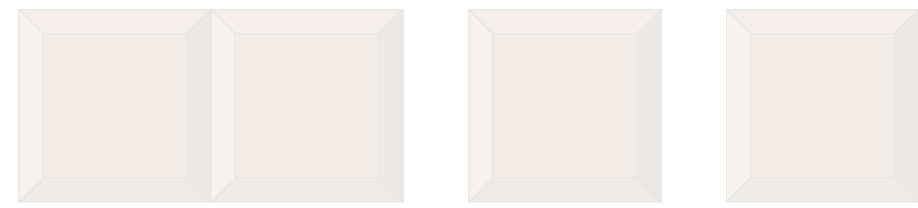
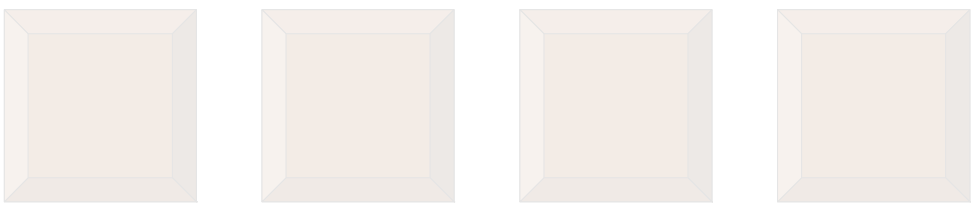
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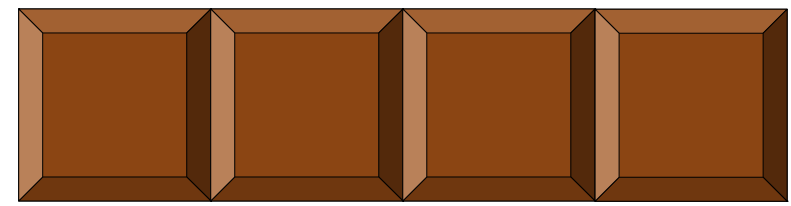
If you eat three pieces first, you then eat the remaining 1×1 chocolate bar any way you'd like.



There are eight ways to eat a 1×4 chocolate bar.



Or you could eat the whole chocolate bar at once. Ah, gluttony. 😊



There are eight ways to eat a 1×4 chocolate bar.

Eating a Chocolate Bar

- There's...
 - 1 way to eat a 1×1 chocolate bar,
 - 2 ways to eat a 1×2 chocolate bar,
 - 4 ways to eat a 1×3 chocolate bar, and
 - 8 ways to eat a 1×4 chocolate bar.
- ***Our guess:*** There are 2^{n-1} ways to eat a $1 \times n$ chocolate bar for any natural number $n \geq 1$.
- And we think it has something to do with this insight:
we eat the bar either by
 - eating the whole thing in one bite, or
 - eating some piece of size k , then eating the remaining $n - k$ pieces however we'd like.
- Let's formalize this!

Theorem: For any natural number $n \geq 1$, there are exactly 2^{n-1} ways to eat a $1 \times n$ chocolate bar from left to right.

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As our base case, we prove $P(1)$, that there is exactly $2^{1-1} = 1$ way to eat a 1×1 chocolate bar from left to right.

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There are two options for how to eat the bar. First, we can eat the whole chocolate bar in one bite. Second, we could eat a piece of size r for some $1 \leq r \leq k$, leaving a chocolate bar of size $k+1-r$, then eat that chocolate bar from left to right.

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Summing up this first option, plus all choices of r for the second option, we see that the number of ways to eat the chocolate bar is

$$1 + 2^{k-1} + 2^{k-2} + \dots + 2^2 + 2^1 + 2^0$$

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There are two options for how to eat the bar. First, we can eat the whole chocolate bar in one bite. Second, we could eat a piece of size r for some $1 \leq r \leq k$, leaving a chocolate bar of size $k+1-r$, then eat that chocolate bar from left to right. Since $1 \leq r \leq k$, we know that $1 \leq k+1-r \leq k$, so by our inductive hypothesis there are 2^{k-r} ways to eat the remainder.

Summing up this first option, plus all choices of r for the second option, we see that the number of ways to eat the chocolate bar is

$$1 + 2^{k-1} + 2^{k-2} + \dots + 2^2 + 2^1 + 2^0 = 1 + 2^k - 1 = 2^k.$$

Theorem: For any natural number $n \geq 1$, there are exactly 2^{n-1} ways to eat a $1 \times n$ chocolate bar from left to right.

Proof: Let $P(n)$ be the statement “there are exactly 2^{n-1} ways to eat a $1 \times n$ chocolate bar from left to right.” We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.

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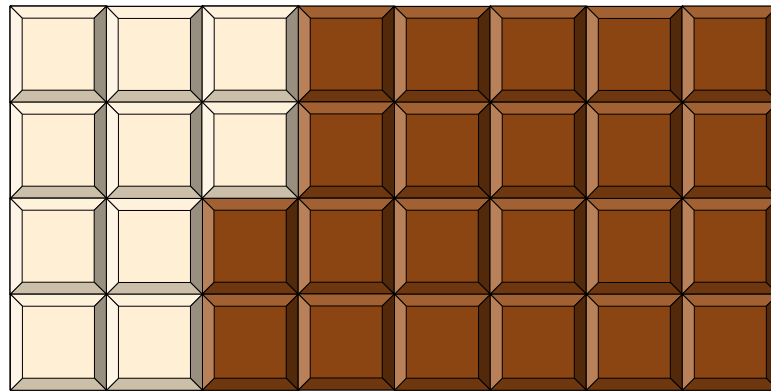
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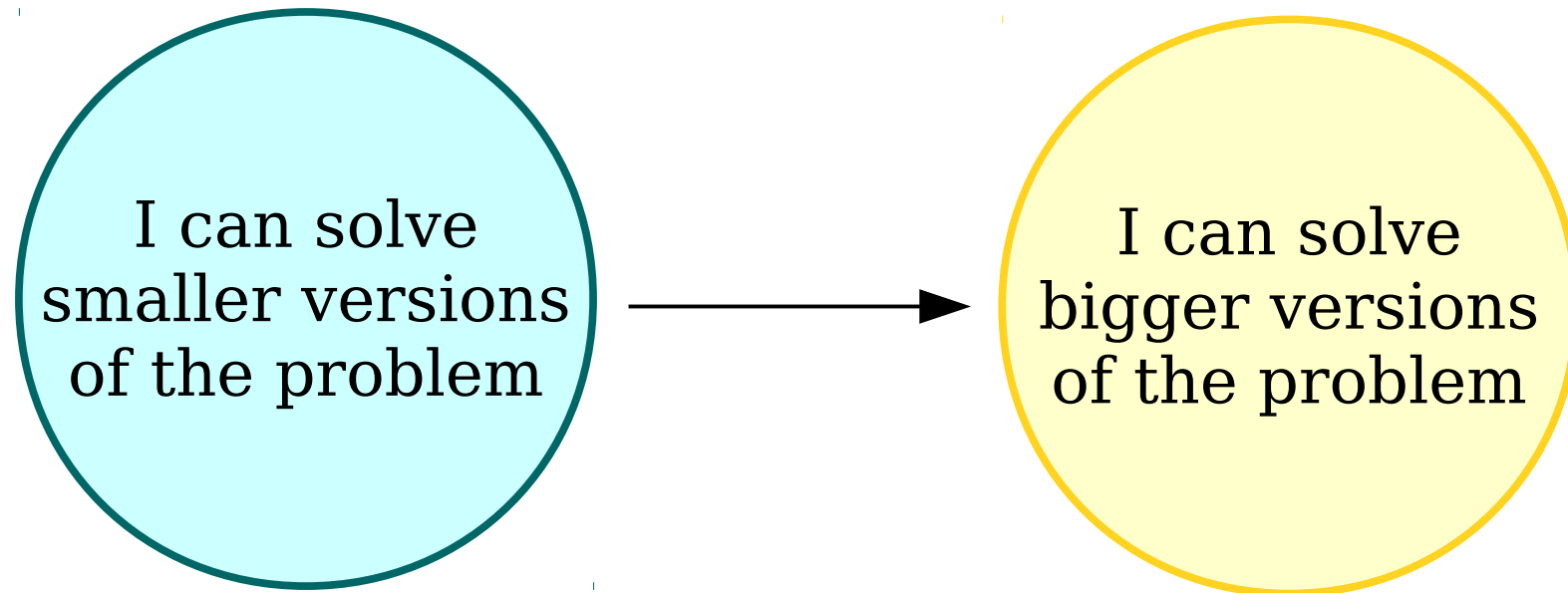
More on Chocolate Bars

- Imagine you have an $m \times n$ chocolate bar. Whenever you eat a square, you have to eat all squares above it and to the left.
- How many ways are there to eat the chocolate bar?

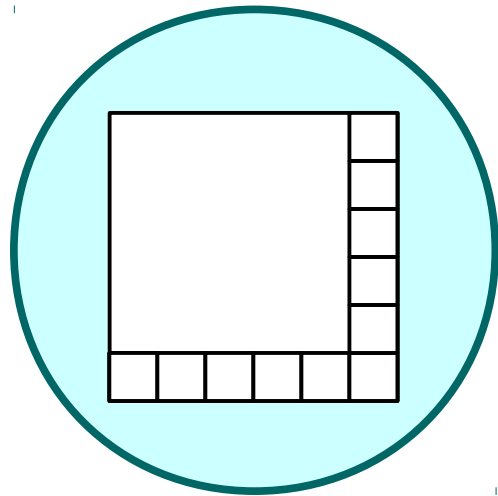


- ***Open Problem:*** Find a non-recursive exact formula for this number, or give an approximation whose error drops to zero as m and n tend toward infinity.

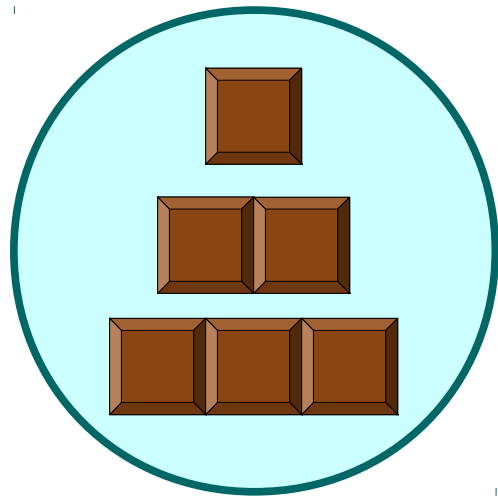
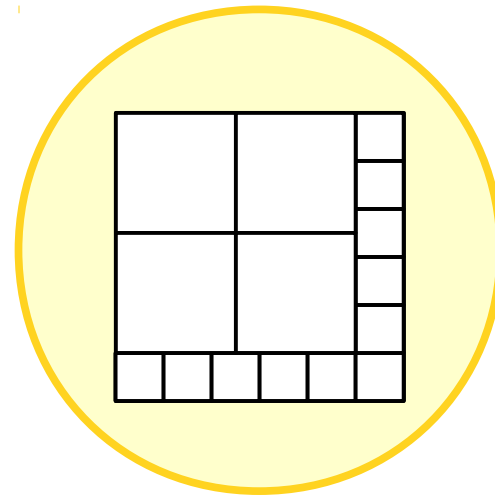
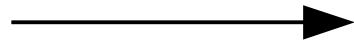
Induction vs. Complete Induction



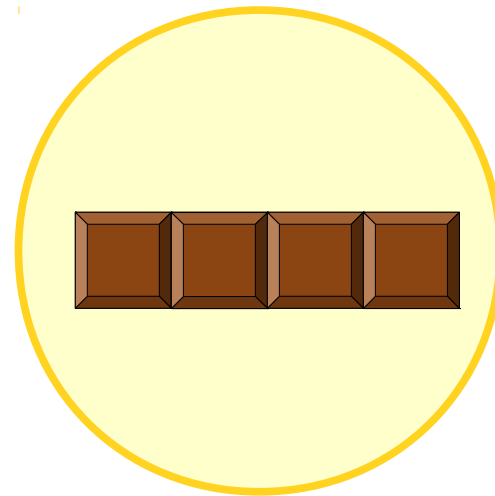
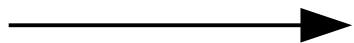
Induction vs. Complete Induction



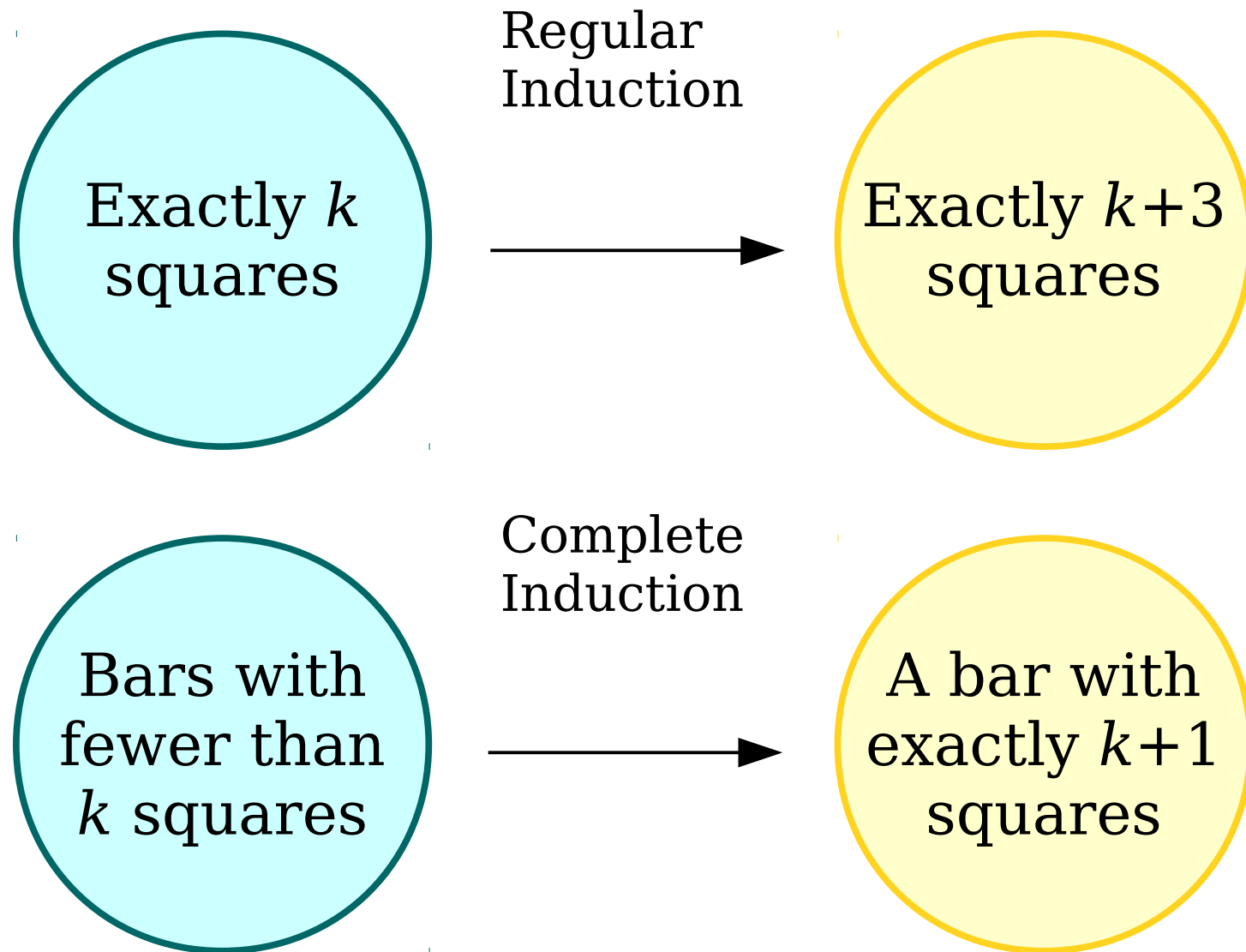
Regular
Induction



Complete
Induction



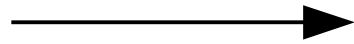
Induction vs. Complete Induction



Induction vs. Complete Induction

Exactly k
squares

Regular
Induction

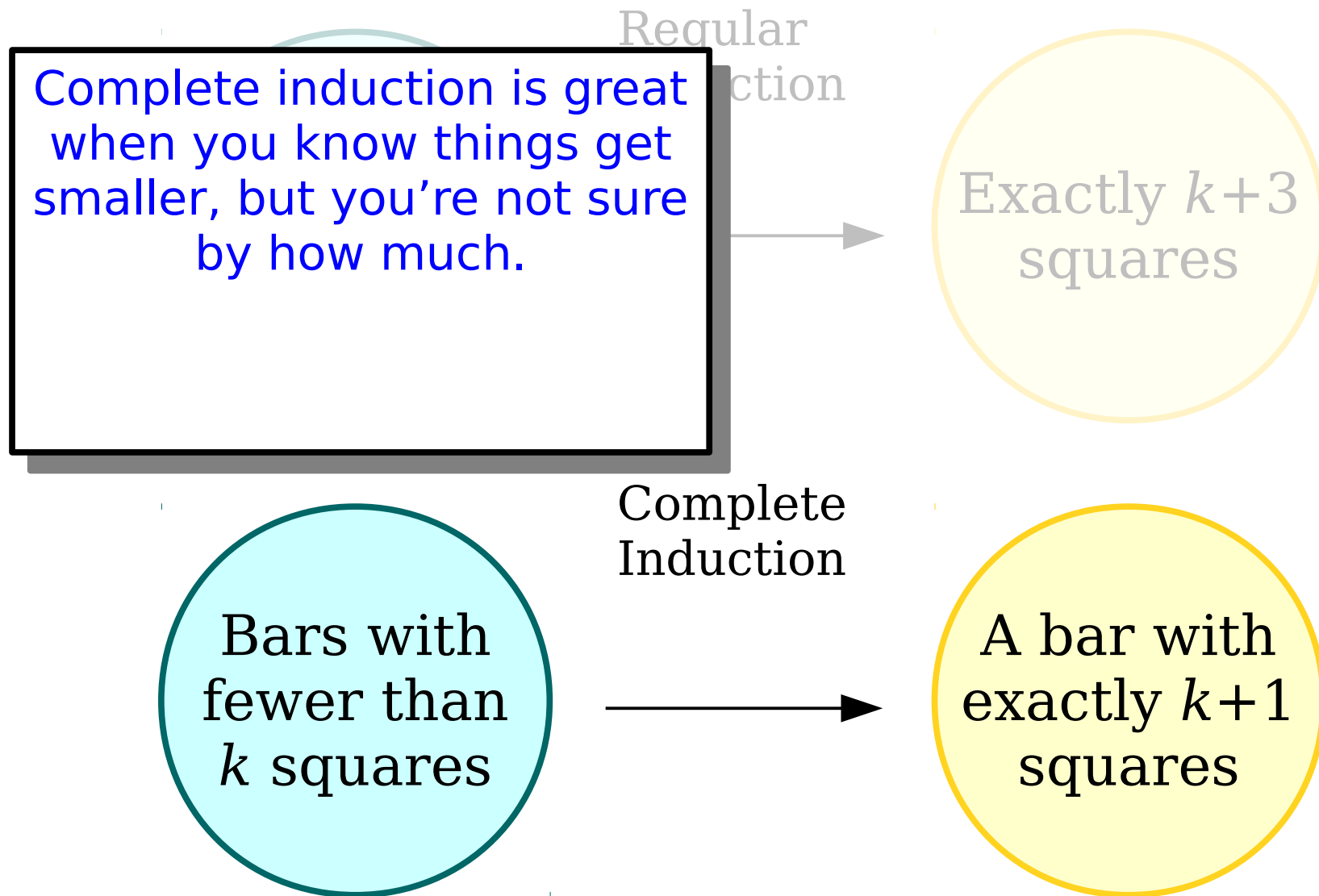


Exactly $k+3$
squares

Bars with
fewer than
 k squares

Regular induction is great
when you know exactly
how much smaller your
“smaller” problem instance
is.

Induction vs. Complete Induction



How Not to Induct

All Horses are the Same Color

$P(n)$ = “All groups of n horses always have the same color”

All Horses are the Same Color

$P(0)$ = “All groups of 0 horses always have the same color”

Vacuously true!

Base case: $n = 0$

All Horses are the Same Color

Assume $P(k)$ = “All groups of k horses always have the same color”



Inductive hypothesis: $n = k$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"



Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

By $P(k)$, these k horses have the same color



Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

By $P(k)$, these k horses have the same color



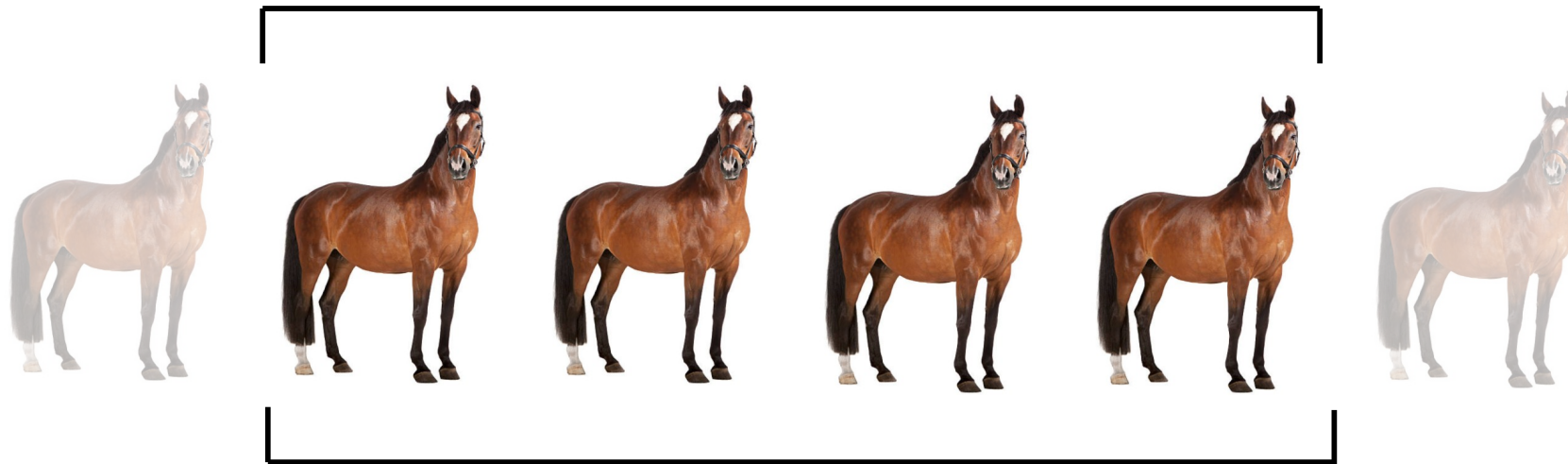
By $P(k)$, these k horses have the same color

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

These horses in the middle were in both sets

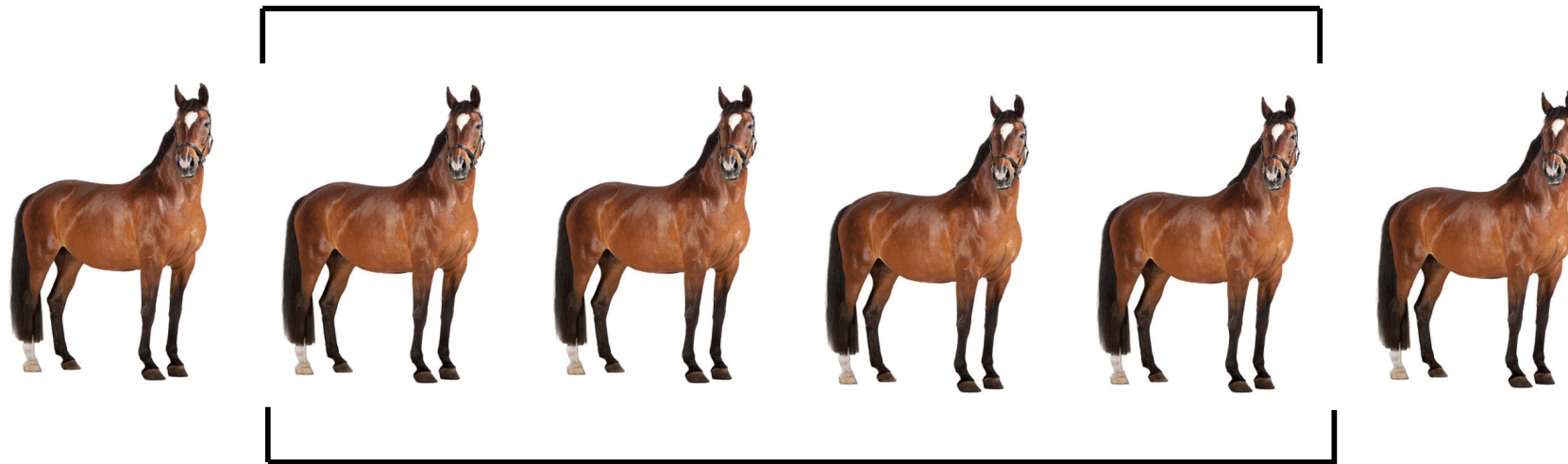


Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

These horses in the middle were in both sets



And we said that both horses on the ends are the same color as these overlapping horses

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"



So all $k+1$ horses have the same color!

Inductive hypothesis: $n = k+1$

⚠ *Incorrect!* ⚠ *Proof:* Let $P(n)$ be the statement “all groups of n horses are the same color.” We will prove by induction that $P(n)$ holds for all natural numbers n , from which the theorem follows.

As our base case, we prove $P(0)$, that all groups of 0 horses are the same color. This statement is vacuously true because there are no horses.

For the inductive step, assume that for an arbitrary natural number k that $P(k)$ is true and that all groups of k horses are the same color. Now consider a group of $k+1$ horses. Exclude the last horse and look only at the first k horses. By the inductive hypothesis, these horses are the same color. Next, exclude the first horse and look only at the last k horses. Again we see by the inductive hypothesis that these horses are the same color.

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color. Thus $P(k+1)$ holds, completing the induction. ■

⚠️ Incorrect! ⚠️ Proof: Let $P(n)$ be the statement “all groups of n horses are the same color.” We will prove by induction that $P(n)$ holds for all natural numbers n , from which the theorem follows.

1

As our base case, we prove $P(0)$, that all groups of 0 horses are the same color. This statement is vacuously true because there are no horses.

2

For the inductive step, assume that for an arbitrary natural number k that $P(k)$ is true and that all groups of k horses are the same color. Now consider a group of $k+1$ horses. Exclude the last horse and look only at the first k horses. By the inductive hypothesis, these horses are the same color. Next, exclude the first horse and look only at the last k horses. Again we see by the inductive hypothesis that these horses are the same color.

3

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color. Thus $P(k+1)$ holds.

4

What's wrong with this proof?
(by paragraph number of the
location of the error)

What's going on here?

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

The horse in the middle is in both sets



Let's look at the $k=2$ case, so that's $k+1=3$ horses.

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

The horse in the middle is in both sets



Let's look at the $k=2$ case, so that's $k+1=3$ horses.

How many horses are in the middle when $k=1$?

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

These horses in the middle were in both sets



When $k=1$, we analyze
 $k + 1 = 2$ horses.

Inductive hypothesis: $n = k+1$

⚠ **Incorrect!** ⚠ **Proof:** Let $P(n)$ be the statement “all groups of n horses are the same color.” We will prove by induction that $P(n)$ holds for all natural numbers n , from which the theorem follows.

As our base case, we prove $P(0)$, that all groups of 0 horses are the same color. This statement is vacuously true because there are no horses.

For the inductive step, assume that $P(k)$ is true for some natural number k . Consider a group of $k+1$ horses. Exclude the first k horses. The remaining horses are the same color. Next, exclude the last k horses. Again, the remaining horses are the same color.

The logic in our inductive step does not allow us to get from $P(1)$ to $P(2)$.

Specifically, there are no non-excluded horses that were in both sets.

the same color. Now, exclude the first k horses and look only at the last k horses. These horses are the same color.

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color. Thus $P(k+1)$ holds, completing the induction. ■

Non-Issues with this Proof

- *“We should have proven additional base cases”*
 - A proof by induction only needs a single base case, so the fact that we only have one here is not in itself an issue.
- *“We should have used complete induction”*
 - Complete induction wouldn't have helped us here either, since our inductive step would still need to use $P(0)$ and $P(1)$ to prove $P(2)$.

Induction Debugging Tips

- Remember that induction requires two parts: the base case and the inductive step
- If you see an induction proof of a false statement, one of these pieces must be broken
- Recommendation: try playing the induction out one step at a time (Is the base case true? From the base case, does the reasoning in your inductive step allow you to conclude the next statement? What about the following statement? The one after that? etc.)

An Important Milestone

Recap: *Discrete Mathematics*

- The past five weeks have focused exclusively on discrete mathematics:

Induction

Functions

Graphs

The Pigeonhole Principle

Formal Proofs

Mathematical Logic

Set Theory

Cardinality

- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

Next Up: *Computability Theory*

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
 - How do we model computation itself?
 - What exactly is a computing device?
 - What problems can be solved by computers?
 - What problems *can't* be solved by computers?
- ***Get ready to explore the boundaries of what computers could ever be made to do.***

Next Time

- ***Formal Language Theory***
 - How are we going to formally model computation?
- ***Finite Automata***
 - A simple but powerful computing device made entirely of math!
- ***DFAs***
 - A fundamental building block in computing.